Name ____

Math 425 - Midterm 2
April 7, 2020.

You are expected to uphold the Code of Academic Integrity. I certify that all of the work on this test is my own.

Signature: ______________________________

The exam is open book. Correct answers without proper justification will not receive full credit. Clearly highlight your answers and the steps taken to arrive at them: illegible work will not be graded. **Solve each problem in a separate page, and always indicate the number and part of the problem you are solving.** When you finish, scan all your work and **upload a PDF to Canvas** (you may use CamScanner app). Please **do not upload photos**.

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Problem	Points	Your score
1	25	
2	25	
3	25	
4	25	
Total	100	

Problem 1 [25 points]

The movement of a rope fixed on one end and free to move on the other is modeled by a damped wave equation with Dirichlet-Neumann boundary conditions:

$$u_{tt} = u_{xx} - 3u_t, \;\; 0 < x < \pi,$$
$$u(0,t) = 0, \;\; u_x(\pi, t) = 0.$$

Part a. [4 points] Consider a product solution $u(x,t) = \phi(x)h(t)$. Show that $\phi(x)$ and $h(t)$ must satisfy the following equations

$$\phi''(x) = -\lambda\phi(x), \;\; \phi(0) = \phi'(\pi), \;\; 0 < x < \pi,$$

$$h''(t) + 3h'(t) = -\lambda h(t), \;\; t > 0.$$

Part b. [4 points] *Without solving* any of the equations on Part a., show that the eigenvalues λ must be strictly positive, and that eigenfunctions ϕ corresponding to different eigenvalues must be orthogonal.

Part c. [4 points] Compute the eigenfunctions ϕ_n and the eigenvalues λ_n.

Part d. [4 points] For each λ_n found in Part c., solve the time ODE for $h(t)$.

Part e. [4 points] Find a solution $u(x,t)$ to the problem that is decreasing in time for all values of x in $(0, \pi)$.

Part f. [5 points] Find the candidate solution to the problem with initial conditions

$$u(x,0) = 2\pi x - x^2, \;\;\; u_t(x,0) = 0.$$

Part g. [Optional, no points] The candidate solution in Part f. should be given by a series. Is this series an actual solution of the PDE $u_{tt} = u_{xx} - 3u_t$? Notice that term by term differentiation is not generally allowed (check pages 44, 45 of W. Strauss book to see why it is in fact a solution of the PDE). Show that the series at $t = 0$ is uniformly convergent to the initial data $2\pi x - x^2$. Finally, show that this solution is the unique solution to the problem (notice that in general the method of separation of variables only proposes a particular type of solution)

Problem 2 [25 points]

Consider the function

$$f(x) = \frac{1}{|1-x|^{2/5}}, \quad 0 < x < 2,$$

and its Fourier cosine series in $(0,2)$, which we denote by

$$\mathcal{C}(f)(x) = \sum_{n=0}^{\infty} c_n \phi_n(x), \quad \phi_n(x) = \cos\left(\frac{n\pi x}{2}\right).$$

We use a computer to find the first ten coefficients of this expansion:

$$c_0 = 5/3, \;\; c_2 \approx -0.948, \;\; c_4 \approx 0.562, \;\; c_6 \approx -0.464, \;\; c_8 \approx 0.378, \;\; c_{10} \approx -0.339,$$

$$c_1 = 0, \;\; c_3 = 0, \;\; c_5 = 0, \;\; c_7 = 0, \;\; c_9 = 0.$$

Notice that this is an example of a function which is not piecewise continuous.

Part a. [5 points] From the above results, we suspect that all the odd coefficients are zero. Prove it:

$$c_{2n-1} = 0, \;\; n \geq 1.$$

Hint: plot the functions $f(x)$ and $\phi_n(x)$ on $(0,2)$. What kind of symmetries do they have across $x = 1$?

Part b. [5 points] Show that the function $f(x)$ is in $L^2(0,2)$ with norm $\|f\|_{L^2} = \sqrt{10}$.

Part c. [5 points] Compute $\sum_{n=0}^{\infty} |c_n|^2$.

Part d. [5 points] Explain why $\sum_{n=0}^{\infty} |c_n| = +\infty$.

Part e. [5 points] Let $\mathcal{S}_6(f)(x) = \sum_{n=0}^{6} c_n \phi_n(x)$ be the approximation of $f(x)$ with seven terms of the series. Find the % error of this approximation, that is, compute the following percentage

$$100 \frac{\|f - \mathcal{S}_6(f)\|_{L^2}}{\|f\|_{L^2}}.$$

What if we use 9 terms? And with 11? You should find very small improvements as N increases: that is due to the lack of smoothness of $f(x)$ (see Part d.)

Problem 3 [25 points]

Consider the eigenvalue problem

$$\phi''(x) + \lambda\phi(x) = 0, \;\; 0 < x < \pi,$$
$$\phi'(0) = \phi(\pi) = 0,$$

which gives

$$\lambda_n = \left(\frac{2n-1}{2}\right)^2, \quad \phi_n(x) = \cos\left(\frac{2n-1}{2}x\right), \;\; n \geq 1,$$

Let f, f' be continuous functions on $[0, \pi]$. Consider its Fourier series $\sum_{n=1}^{\infty} c_n\phi_n$. We want to give a proof that this Fourier series converges pointwise to $f(x)$ for $x \in (0, \pi)$.

Remark: Notice that we cannot prove uniform convergence because we don't know if f satisfies the boundary conditions $f'(0) = 0, f(\pi) = 0$ (you can check in Problem 4 of practice midterm these conditions were needed).

Part a. [5 points] Find the formula for the coefficients c_n.

Part b. [5 points] Denote the partial sums by $\mathcal{S}_N(f)(x) = \sum_{n=1}^{N} c_n \cos\left(\frac{2n-1}{2}x\right)$. Show that they can be written as follows:

$$\mathcal{S}_N(f)(x) = \frac{1}{\pi}\int_0^{\pi} \big(k_N(x-y) + k_N(x+y)\big) f(y)dy,$$

with

$$k_N(x) = \sum_{n=1}^{N} \cos\left(\frac{2n-1}{2}x\right).$$

Part c. [5 points] Show that the kernel k_N can be summed to obtain the expression

$$k_N(x) = \frac{1}{2}\frac{\sin(Nx)}{\sin(x/2)}.$$

Then, after changing variables and using a periodic extension, we can write

$$\mathcal{S}_N(f)(x) = \frac{1}{\pi}\int_0^{\pi} k_N(y) f(x-y)dy + \frac{1}{\pi}\int_0^{\pi} k_N(y) f(x+y)dy. \tag{1}$$

Part d. [5 points] It can be proved that this kernel has *almost* constant average equal to 1/2. More specifically,

$$\frac{1}{\pi}\int_0^{\pi} k_N(y)dy = \frac{1}{2} + a_N, \;\; \lim_{N\to\infty} a_N = 0.$$

Use expression (1) and this property to get

$$\mathcal{S}_N(f)(x) - f(x) = \frac{1}{\pi}\int_0^\pi k_N(y)(f(x-y)-f(x))dy + \frac{1}{\pi}\int_0^\pi k_N(y)(f(x+y)-f(x))dy + 2a_N f(x).$$

Part e. [5 points] Use the last expression in Part d. to complete the proof of pointwise convergence by showing that

$$\lim_{N\to\infty} |S_N(f)(x) - f(x)| = 0, \;\; 0 < x < \pi.$$

Part f. [Optional, no points] Which of the previous steps fails at $x = \pi$ if $f(\pi) \neq 0$?

Remark: We could also have proved this result by calling to the theorem of pointwise convergence of classical Fourier series. Take the function f on $(0, \pi)$ and construct its odd extension around $x = \pi$ on $(0, 2\pi)$, $f_{odd,\pi}$. Then, construct the even extension of this new function around $x = 0$ on $(-2\pi, 2\pi)$, $(f_{odd,\pi})_{even}$. The classical Fourier series of $(f_{odd,\pi})_{even}$ coincides with the Fourier series for f considered in this exercise (you can check it by computing the coefficients and finding which ones vanish).

Problem 4 [25 points]

Consider the eigenvalue problem

$$\phi''(x) + \lambda\phi(x) = 0, \;\; 0 < x < \pi,$$
$$\text{symmetric BC.}$$

We want to prove the uniform convergence of the Fourier series for functions in $C^2[0,\pi]$ and which satisfy the symmetric boundary conditions of the problem ($f \in C^2[0,\pi]$ means that f, f', and f'' exist and are continuous on $[0,\pi]$,)

We will need two important results (we cannot cover them in this course; see W. Strauss chapter 11):

1. There is a smallest eigenvalue, but not a largest: $\lambda_1 < \lambda_2 < \dots,$
2. Asymptotic formula: $\lim_{n\to\infty} \frac{\lambda_n}{n^2} = 1$. That is, $\lambda_n \sim n^2$ for large n.

Part a. [5 points] Denote c_n and d_n the coefficients of the Fourier series of f and f'':

$$f(x) \sim \sum_{n=1}^{\infty} c_n\phi_n(x), \;\; f''(x) \sim \sum_{n=1}^{\infty} d_n\phi_n(x).$$

Using integration by parts, find the following relationship

$$d_n = -\lambda_n c_n.$$

Part b. [5 points] Show that f'' is in $L^2[0,\pi]$. Which inequality can you find in relation to the constants d_n?

Part c. [5 points] Using Part b., show that

$$\sum_{n\geq N} |c_n| \leq \Big(\sum_{n\geq N} \frac{1}{\lambda_n^2}\Big)^{1/2}\Big(\sum_{n\geq N} d_n^2\Big)^{1/2}.$$

Part d. [5 points] Since there is a smallest eigenvalue and $\lambda_n \to +\infty$, there can only be, if any, a finite amount of negative or zero eigenvalues. So, for large enough n, we can assume that the eigenfunctions are $\sin(\sqrt{\lambda_n}x)$, $\cos(\sqrt{\lambda_n}x)$. Therefore,

$$|\phi_n(x)| \leq 1, \;\; \|\phi_n\|_{L^2}^2 = \pi/2, \quad n \geq N,$$

for some number N.

Show that $\sum_{n=1}^{\infty} c_n\phi_n(x)$ converges uniformly in $(0,\pi)$.

Part e. [5 points] Conclude that $\sum_{n=1}^{\infty} c_n\phi_n(x)$ converges uniformly to $f(x)$ on $(0,\pi)$.