Name ____

Math 241 - Section 002 - Midterm 2
Thursday, April 2, 2020.

You are expected to uphold the Code of Academic Integrity. I certify that all of the work on this test is my own.

Signature: ______________________________

The exam is open book. Correct answers without proper justification will not receive full credit. Clearly highlight your answers and the steps taken to arrive at them: illegible work will not be graded. **Solve each problem in a separate page, and always indicate the number and part of the problem you are solving.** When you finish, scan all your work and **upload a PDF to Canvas** (you may use CamScanner app). Please **do not upload photos**.

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Problem	Points	Your score
1	25	
2	30	
3	30	
4	15	
Total	100	

Problem 1 [25 points]

Consider the following two-dimensional eigenvalue problem with a Robin boundary condition:

$$\Delta\phi + \lambda\phi = 0, \quad \text{in } \Omega,$$
$$\vec{n}\cdot\nabla\phi = \phi \quad \text{on } \partial\Omega,$$

where Ω is a disk of radius R centered at the origin.

Part a. [5 points] Write the Rayleigh quotient for this problem. Briefly discuss if you can deduce the sign of the eigenvalues from it.

Part b. [5 points] Find the general solution to the following ordinary differential equation:

$$x^2 f''(x) + x f'(x) - n^2 f(x) = 0,$$

for any $n \geq 0$.

Part c. [8 points] Let $\phi(r,\theta) = F(r)G(\theta)$. When $\lambda = 0$, write the ODEs that F and G satisfy, together with their respective boundary conditions (two for each). You may assume that we are only looking for physically relevant solutions.

Part d. [7 points] For which values of the radius R, if any, is $\lambda = 0$ an eigenvalue?

Problem 2 [30 points]

Consider a bounded domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$. The eigenfunctions of the following eigenvalue problem

$$
\begin{aligned}
4\phi_{xx} + \phi_{yy} &= \lambda\phi, \quad \text{in } \Omega, \\
\phi &= 0, \quad \text{on } \partial\Omega,
\end{aligned}
$$

are given by

$$
\phi_{n,m}(x,y) = \sin\left(\frac{n\pi x}{2}\right)\sin(m\pi y) - \sin\left(\frac{m\pi x}{2}\right)\sin(n\pi y), \tag{1}
$$

where $n = 1, 2, \ldots,$ and $m = 1, 2, \ldots$.

Part a. [7 points] Compute the eigenvalue $\lambda_{n,m}$ corresponding to the eigenfunction $\phi_{n,m}$.

Part b. [14 points] Write the general solution to the wave equation

$$
\begin{aligned}
u_{tt} &= 4u_{xx} + u_{yy}, \quad \text{in } T, \\
u &= 0, \quad \text{on } \partial T.
\end{aligned}
$$

Part c. [9 points] Write formulas for the coefficients in the general solution found in Part b. to match given initial conditions $u(x,y,0) = f(x,y)$ and $u_t(x,y,0) = g(x,y)$. You may use that $\{\phi_{n,m}\}$ are orthogonal, with $\int_\Omega \phi_{n,m}^2(x,y)dA = C$ a given constant.

Problem 3 [30 points]

Part a. [10 points] Convert the following boundary value problem to Sturm-Liouville form:

$$\phi'' + (2 - 4x)\phi' = -\lambda\phi,$$

$$\phi(0) = \phi(1) = 0.$$

Part b. [10 points] For the problem in Part a., show that $\phi(x) = x(1-x)$ is an eigenfunction, and find its eigenvalue. Then briefly explain why this is the lowest eigenvalue of this problem.

Part c. [5 points] The eigenvalues of a certain eigenvalue problem are given by the equation $\sinh\lambda \sin\lambda = 1$. Can this eigenvalue problem be a regular Sturm-Liouville one? Explain why or why not. Hint: Make a plot of $\frac{1}{\sinh(x)}$.

Part d. [5 points] The eigenvalues of a certain eigenvalue problem are given by $\lambda_n = n^2$, $n \geq 1$, and the corresponding eigenfunctions by $\phi_n(x) = \cos(\lambda_n x/2)$ for $0 < x < 1$. Can this eigenvalue problem be a regular Sturm-Liouville one? Explain why or why not.

Problem 4 [15 points]

Consider a membrane with annulus shape Ω. Its inner circle $\partial\Omega_1$ has radius a and is fixed, while the outer one $\partial\Omega_2$ has radius b and is free to move:

$$u_{tt} = \Delta u,$$

subject to the boundary and initial conditions

$$u|_{\partial\Omega_1} = 0, \;\; \vec{n}\cdot\nabla u|_{\partial\Omega_2} = 0, \;\; u|_{t=0} = f, \;\; u_t|_{t=0} = 0.$$

You may assume that separation of the time and spatial variables yields the following equations

$$u(r,\theta,t) = \phi(r,\theta)h(t),$$
$$h''(t) = -\lambda h(t),$$
$$\Delta\phi + \lambda\phi = 0.$$

Furthermore, you may assume that separating the two spatial variables as

$$\phi(r,\theta) = F(r)G(\theta)$$

yields the solutions

$$G(\theta) = \cos(n\theta), \sin(n\theta), \;\; n \geq 0,$$

and the equation

$$r\frac{d}{dr}\big(rF'(r)\big) + (\lambda r^2 - n^2)F(r) = 0, \;\; n \geq 0$$

in the variable r.

Part a. [3 points] Write the boundary conditions for the radial part $F(r)$.

Part b. [4 points] Show that all the eigenvalues λ are positive $\lambda > 0$.

Part c. [4 points] Find an equation that determines the eigenvalues λ.

Part d. [4 points] Write out the general form of the solution. You may assume that for each n, there are infinitely many eigenvalues $\lambda_{n,m}$.