Name ____

Math 241 - Section 002 - Midterm 2 - Practice exam
Thursday, April 2, 2020.

You are expected to uphold the Code of Academic Integrity. I certify that all of the work on this test is my own.

Signature: ______________________________

The exam is open book. Correct answers without proper justification will not receive full credit. Clearly highlight your answers and the steps taken to arrive at them: illegible work will not be graded. **Solve each problem in a separate page, and always indicate the number and part of the problem you are solving.** When you finish, scan all your work and **upload a PDF to Canvas** (you may use CamScanner app). Please **do not upload photos**.

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Problem	Points	Your score
1	25	
2	25	
3	25	
4	25	
Total	100	

Problem 1 [25 points]

Part a. and Part b. of this problem are independent of each other.

Part a. [10 points] Suppose that in a three-dimensional region Ω

$$\Delta\phi = f,$$

with f given and $\vec{n} \cdot \nabla\phi = 0$ on the boundary $\partial\Omega$.

a.1) Show mathematically that, if there is a solution, then

$$\int_{\Omega} f dV = 0.$$

b.2) Briefly explain physically why condition in Part a. must hold for a solution to exist.

Part b. [15 points] Consider the following two-dimensional eigenvalue problem in a square with mixed boundary conditions (Dirichlet and Robin):

$$\begin{aligned} \Delta\phi + \lambda\phi = 0, \quad &\text{in } \Omega, \\ \vec{n} \cdot \nabla\phi = \phi \quad &\text{on } \partial\Omega_1, \\ \phi = 0 \quad &\text{on } \partial\Omega_2, \end{aligned}$$

where Ω is the square $0 < x < 1$, $0 < y < 1$, $\partial\Omega_1$ are the vertical sides $x = 0, x = 1$, $0 < y < 1$, and $\partial\Omega_2$ are the horizontal sides $y = 0, y = 1$, $0 < x < 1$. Remember that $\vec{n}$ denotes the outward unit normal vector to the boundary.

b.1) Write the problem, including boundary conditions, in the variables x and y.

b.2) Determine whether $\lambda = 0$ is an eigenvalue or not.

Problem 2 [25 points]

A metal plate of square shape $L = \pi$, $H = \pi$, is fixed on two sides and free two move on the other two sides. Its inner friction makes it move as a damped membrane:

$$u_{tt}(x, y, t) = \Delta u(x, y, t) - \alpha u_t(x, y, t), \;\; 0 < x < \pi, 0 < y < \pi,$$

$$u_x(0, y, t) = u_x(\pi, y, t) = 0, \;\; u(x, 0, t) = u(x, \pi, t) = 0.$$

Part a. [8 points] Define the energy $E(t) = \displaystyle\int_\Omega \left(|u_t|^2 + |\nabla u|^2\right)$. Show that for $\alpha > 0$, as far as the membrane is moving, $E(t)$ it is a strictly decreasing function.

Part b. [9 points] For $\alpha = \sqrt{5}$, find the general solution to the problem and use it to compute $\lim_{t\to\infty} u(x, y, t)$. Briefly relate this results with Part a.

Part c. [8 points] Find the coefficients of your solution in Part b. if the plate is initially at equilibrium position $u(, y, 0) = 0$ but with initial velocity $u_t(x, y, 0) = f(x, y)$.

Problem 3 [25 points]

Consider the fourth-order eigenvalue problem

$$\frac{d^4\phi}{dx^4} + \lambda e^x \phi = 0, \quad 0 < x < 1,$$

with boundary conditions given by

$$\phi(0) = 0, \quad \phi(1) = 0, \quad \phi'(0) = 0, \quad \phi''(1) = 0.$$

Use integration by parts when needed in the following questions.

Part a. [6 points] Show that eigenfunctions corresponding to different eigenvalues are orthogonal (with respect to a certain weight).

Part b. [7 points] Find a *Rayleigh quotient.* That is, given an eigenfunction ϕ, find a formula that determines the corresponding eigenvalue. Use it to show that $\lambda \leq 0$.

Part c. [6 points] Is $\lambda = 0$ an eigenvalue?

Part d. [6 points] Assuming the eigenfunctions form a complete set, find the coefficients of the following expansion

$$\sin x = \sum_{\lambda \leq 0} A_\lambda \phi_\lambda(x), \quad 0 < x < 1.$$

Problem 4 [25 points]

Consider the wave equation

$$u_{tt} = \Delta u$$

on a disk Ω of radius R subject to the boundary and initial conditions

$$u(R, \theta, t) = -u_r(R, \theta, t), \;\; u(r, \theta, 0) = f(r, \theta), \;\; u_t(r, \theta) = 0.$$

You may assume that separation of the time and spatial variables yields the following equations

$$u(r, \theta, t) = \phi(r, \theta) h(t),$$
$$h''(t) = -\lambda h(t),$$
$$\Delta\phi + \lambda\phi = 0.$$

Furthermore, you may assume that separating the two spatial variables as

$$\phi(r, \theta) = F(r) G(\theta)$$

yields the solutions

$$G(\theta) = \cos(n\theta), \sin(n\theta), \;\; n \geq 0,$$

and the equation

$$r\frac{d}{dr}\big(rF'(r)\big) + (\lambda r^2 - n^2) F(r) = 0, \;\; n \geq 0$$

in the variable r.

Part a. [5 points] Rewrite the boundary condition for $\phi(r, \theta)$ in the form $c_1\phi + c_2\vec{n}\cdot\nabla\phi = 0$.

Part b. [5 points] Show that all the eigenvalues λ are positive $\lambda > 0$. You may assume that c_1 and c_2 in Part a. are positive.

Part c. [5 points] What is the solution to the r-dependent problem, for each value of $n \geq 0$? Write down an equation that λ and F must satisfy.

Part d. [5 points] Write out the general form of the solution. You may assume that for each n, there are infinitely many eigenvalues $\lambda_{n,m}$.

Part e. [5 points] Make a reasonable assumption about orthogonality (no need to prove it) to solve for the coefficients in your answer to Part d.