Theorem 3: $f \in C^1(\mathbb{R})$ periodic 2π $\Rightarrow$ classical Fourier series converge to $f(x)$ $\forall x \in \mathbb{R}$.

Proof:

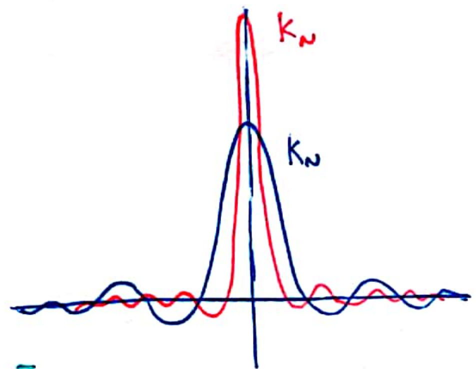
- Goal: Show that $\lim_{N\to\infty} |S_N(f)(x) - f(x)| = 0$, where $S_N(f)(x) = \sum_{n=0}^{N} \left(a_n \cos(nx) + b_n \sin(nx)\right)$.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\cos(ny)\,dy, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\sin(ny)\,dy.$$

- Steps:

1) Lemma: $S_N(f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(y) f(x-y)\,dy$,

$$K_N(x) = 1 + 2\sum_{n=1}^{N} \cos(nx) = \frac{\sin\left((2N+1)\frac{x}{2}\right)}{\sin(x/2)}$$



$$\boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(y)\,dy = 1 \quad \forall N}$$

Proof: Pages 3, 4, 5.

3] Conclussion

$$S_N(f)(x) - f(x) = \frac{1}{2\pi} \underbrace{\left\langle \sin\left(\left(N+\tfrac{1}{2}\right)y\right), \frac{f(x-y)-f(x)}{\sin(y/2)} \right\rangle}_{\substack{N^{th} \text{ coefficient of Fourier expansion of} \\ g(y) \text{ on the set } \{\sin((N+\frac{1}{2})y)\}}} \xrightarrow{(N\to+\infty)} 0$$

$$g(y) \sim \sum_{N=1}^{\infty} c_N \phi_N(y) \quad \text{with} \quad \phi_N(y) = \sin\left(\left(N+\tfrac{1}{2}\right)y\right), \quad c_N = \frac{\langle \phi_N, g \rangle}{\|\phi_N\|_{L^2}^2}$$

Bessel's inequality:

$$\sum_{N=1}^{\infty} c_N^2 \|\phi_N\|_{L^2}^2 \leq \|g\|_{L^2}^2 < +\infty. \Rightarrow c_N \xrightarrow{(N\to+\infty)} 0 \;//.$$

2) Convergence: L^2 theory + $f \in C^1$

$$\to S_N(f)(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(y)\big(f(x-y) - f(x)\big)\,dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\sin\left(\left(N+\tfrac{1}{2}\right)y\right)\frac{f(x-y)-f(x)}{\sin(y/2)}}_{\langle \dots, \dots \rangle}\,dy$$

Proposition:

1) $\{\sin((n+\frac{1}{2})y)\}_{n\geq 1}$ is an orthogonal set. "Proof": $\left.\begin{array}{l}\phi'' + \lambda\phi = 0,\ 0<x<\pi \\ \phi(0) = 0 = \phi'(\pi)\end{array}\right\}$

2) For each x (parameter), and $f \in C^1$, the function

$$g(y) = \frac{f(x-y)-f(x)}{\sin(y/2)}$$ is in $L^2(-\pi,\pi)$. Proof. $\|g\|_{L^2}^2 = \int_{-\pi}^{\pi} \frac{|f(x-y)-f(x)|^2}{|\sin(y/2)|^2}\,dy =$

$$= \int_{-\pi}^{\pi} \frac{|f(x-y)-f(y)|^2}{|y/2|^2}\,\frac{|y/2|^2}{|\sin(y/2)|^2}\,dy < +\infty.$$