Math 241: Lecture 14

Higher Dimensional PDEs II (Helmholtz Equation and Generalities)

Contents

1. Introduction

A brief big picture of what we have learned so far in the course can be summarized in the following points

Lecture 2: We introduced two physical phenomena (heat diffusion and waves) in a 1d setting and used mathematics to model them as PDEs with certain boundary conditions.

Lecture 3: We found that the time-independent (equilibrium) solutions of those models were easier to compute (this corresponds to solving the Laplace equation). In particular, for the heat equation in certain situations (when all the BCs were of Neumann type, that is, heat fluxes), we needed to relate the initial distribution with the equilibrium one using the conservation of energy, which we were able to deduce mathematically from the equation.

Lec. 4-12: We learned the method of separation of variables to solve the previous (time-dependent) models. The method works for much more general problems. In general, one can try it for *linear* and *homogeneous* PDEs with linear and homogeneous BCs.

- To be able to consider general initial conditions, the set of eigenfunctions has to be complete. Fourier Series theorem guarantees this property when the eigenfunctions are sines and cosines.
- To be able to find the coefficients of these Fourier Series, the eigenfunctions must be orthogonal. We could check it explicitly for sines and cosines.
- Later, we realize that we could only solve a few problems explicitly. As soon as the coefficients are not constant (for example, if the physical properties depend on the temperature, etc.) we cannot explicitly solve the eigenvalue problem. However, Sturm-Liouville theory ensures that for the so-called *regular Sturm-Liouville problems* the crucial properties still hold: real eigenvalues, existence of a smallest one but not a largest one, completeness of the set of eigenfunctions, and orthogonality of the eigenfunctions. In summary, we can solve the problems in terms of the eigenvalues and eigenfunctions (these must be computed numerically).

Lectures 13: We started a brief introduction to PDEs in higher in dimensions. We want to compute how the temperature evolves in a surface or region of the space, how membranes vibrate or how sound and light propagate. We checked that the procedure of the method of separation of variables worked similarly. When the geometry is simple (a rectangle in this case) we can compute the solution to the heat and wave equation with constant coefficients in two and three dimensions explicitly (we need two (resp. three) separation constants and the solution will be given as a double (resp. three) infinite sum).

In this lecture, we want first to give results for two and three-dimensional eigenvalue problems analogous to the ones in Sturm-Liouville theory for one-dimensional eigenvalue problems. Then, we will give some examples where the concepts of equilibrium solutions and energy comes to play again.

2. Helmholtz Equation

In the one-dimensional problems we studied, the most typical ODE in the eigenvalue problem was

$$\phi'' + \lambda\phi = 0.$$

We could easily solve it and test the values of λ using the associated boundary conditions. Then, we generalize the results to the Sturm-Lioville ODE

$$\frac{d}{dx}\big(p(x)\phi'(x)\big) + q(x)\phi(x) + \lambda\sigma(x)\phi(x) = 0,$$

under certain conditions on $p(x), q(x), \sigma(x)$ and the boundary conditions.

In two and three dimensions, we found in last lecture that after separating the time and spatial variables (for both the heat and wave equation with constant coefficients) the eigenvalue problem was

$$\begin{aligned} \Delta\phi + \lambda\phi = 0, &\quad \text{in } \Omega, \\ \alpha\phi + \beta\vec{n}\cdot\nabla\phi = 0, &\quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$, and $\vec{n}$ is the outward unit normal vector to the boundary (we are assuming Dirichlet, Neumann, or Robin BCs). The above equation $\Delta\phi+\lambda\phi = 0$ is called the **Helmholtz equation**. Notice that, as opposed to the 1d case $\phi''+\lambda\phi = 0$, now we cannot explicitly solve this problem for general Ω. So, to keep it simple, we give all the properties that hold for this problem (we will assume that the boundary of the domain is smooth). We state the results in 2d (analogous in 3d):

1. All the eigenvalues are real.

2. There exists a smallest eigenvalue λ_1, an infinite number of eigenvalues, and no largest eigenvalue.

3. The eigenfunctions form a complete set: any piecewise smooth function can be written as a generalized Fourier series

$$f(x, y) \sim \sum_{\lambda} A_\lambda \phi_\lambda(x, y),$$

with the coefficients A_λ appropriately chosen (see point 4). The sum above indicates summation over all the eigenvalues (it will be a double infinite sum).

4. Eigenfunctions corresponding to different eigenvalues are orthogonal over the domain,

$$\int_\Omega \phi_{\lambda_1}\phi_{\lambda_2} dA = 0 \text{ if } \lambda_1 \neq \lambda_2.$$

   We will generally use the notation $\int_\Omega$ to denote integral over a domain (surface in 2d, volume in 3d) and dA, dV for the correspondent differential. For clarity, in the boundary integral $\int_{\partial\Omega}$ (line integral in 2d, surface integral in 3d) we will write dl and dS for the line and surface differential.

5. Rayleigh quotient:

$$\lambda = \frac{-\int_{\partial\Omega} \vec{n}\cdot\nabla\phi dl + \int_\Omega |\nabla\phi|^2 dA}{\int_\Omega \phi^2 dA}.$$

Remark 1. *As opposed to the 1d case, each eigenvalue might have many eigenfunctions. Nevertheless, one can always use Gram-Schmidt algorithm to obtain orthogonal eigenfunctions in this case (as in the Spectral Theorem for symmetric matrices). We will not use this in this course, so don't worry if this remark doesn't make too much sense to you.*

Exercise 2. *Check all this properties for the wave equation exercise done in Lecture 13.*

2.1. Dirichlet and Neumann Laplacian

The following two problems are very important. They are the eigenvalue problems one obtains after separating variables for the (constant coefficients) heat or wave equation with Dirichlet or Neumann boundary conditions. Moreover, notice that when $\lambda = 0$ the problems are the *Dirichlet or Neumann Laplacian* (that is, the Laplace equation with Dirichlet or Neumann boundary conditions).

Problem 3. *Consider the two-dimensional eigenvalue problem*

$$\Delta\phi + \lambda\phi = 0 \;\; \text{in } \Omega \subset \mathbb{R}^2,$$
$$\phi = 0 \;\; \text{on } \partial\Omega,$$

with Ω a bounded, connected, smooth domain.

a) Prove that $\lambda \geq 0$.

b) Is $\lambda = 0$ an eigenvalue? If so, what is an eigenfunction corresponding to $\lambda = 0$?

From a physics perspective, you should expect that the eigenvalues above satisfies $\lambda > 0$: the temperature distribution in a domain with fixed zero temperature on the boundary should evolve to zero everywhere. In other words, for $\lambda = 0$ the solution should be $\phi = 0$ everywhere (and thus not an eigenvalue). We can prove all this mathematically. Also, notice that solutions to the Laplace equation correspond to time-independent solutions of the heat equation.

Problem 4. *Consider the two-dimensional eigenvalue problem*

$$\Delta\phi + \lambda\phi = 0 \;\; \text{in } \Omega \subset \mathbb{R}^2,$$
$$\vec{n}\cdot\nabla\phi = 0 \;\; \text{on } \partial\Omega,$$

with Ω a bounded, connected, smooth domain.

a) Prove that $\lambda \geq 0$.

b) Is $\lambda = 0$ an eigenvalue? If so, what is an eigenfunction corresponding to $\lambda = 0$?

The first part can be shown by a direct application of the Rayleigh quotient. However, that will not suffice to determine if $\lambda = 0$ is an eigenvalue or not. But notice that, by the definition of eigenvalue, if we are able to find a non-zero function ϕ that solves the PDE and BCs for $\lambda = 0$, the $\lambda = 0$ is an eigenvalue. Of course, in general this is a hard method to find if a given number is an eigenvalue: the PDE may be too hard. In this particular case, it is immediate to see that $\phi = c$ for any constant $c \neq 0$ solves the problem (PDE and BCs). Therefore, $\lambda = 0$ is an eigenvalue and a correspondent eigenfunction is any constant.

3. Integration by Parts, Energy, and Equilibrium Solutions

Why should we have expected $\lambda = 0$ to be an eigenvalue in Problem 4? To get some intuition from physics, try to solve the following problem:

Problem 5. *Consider the heat equation with insulation BC*

$$
\begin{aligned}
u_t = k\Delta u \quad &in\ \Omega \subset \mathbb{R}^2, \\
\vec{n} \cdot \nabla u = 0 \quad &on\ \partial\Omega,
\end{aligned}
$$

with Ω a bounded, connected, smooth domain, and initial temperature distribution

$$u(x, y, 0) = f(x, y).$$

What is the equilibrium solution to this problem? That is, towards which temperature distribution does the solution $u(x, y, t)$ converges to as $t \to \infty$?

Hint: You will need a particular case of the integration by parts formula

$$\int_\Omega f \Delta g dA = -\int_\Omega \nabla f \cdot \nabla g dA + \int_{\partial\Omega} f \vec{n} \cdot \nabla g dl.$$

This is itself a particular case of the general one

$$\int_\Omega f \partial_{x_i} g dA = -\int_\Omega \partial_{x_i} f g dA + \int_{\partial\Omega} f g n_i dl,$$

where ∂_{x_i} is the partial derivative with respect to the ith variable, and n_i is the ith component of the unit outward normal vector.

Additional exercises to practice:

Exercise 6. *Consider the Laplace equation $\Delta u = 0$ in a three-dimensional region, where $u(x, y, z)$ is the (time-independent) temperature. Suppose that the heat flux is given on the boundary*

$$
\begin{aligned}
\Delta u = 0 \quad &in\ \Omega \subset \mathbb{R}^3, \\
\vec{n} \cdot \nabla u = f \quad &on\ \partial\Omega.
\end{aligned}
$$

a) *Explain physically why the condition*

$$\int_{\partial\Omega} f dS = 0$$

must hold for a solution to exist.

b) *Show Part a) mathematically.*

Exercise 7. *Suppose* u *solves* $u_t = \Delta u$ *in a bounded domain* $B \subset \mathbb{R}^3$ *subject to*

$$u(x, y, z, 0) = f(x, y, z) \;\; \textit{in } B,$$
$$\vec{n} \cdot \nabla u = g \;\; \textit{on } \partial B.$$

Let $E(t)$ *be the total heat energy,*

$$E(t) = \int_B u(x, y, z, t) dx dy dz.$$

Find the total heat energy at $t = 100$*.*