[Additional example S-L problems]

[Done in Lecture 13]

<u>Exercise:</u> Consider the BVP

$$\left.\begin{aligned} &\phi'' + (3-3x)\phi' + \lambda\phi = 0, \quad x\in[0,2] \\ &\phi(0) = \phi(2) = 0 \end{aligned}\right\}$$

a) Rewrite the ODE in Sturm-Liouville form.

Multiply by an integrating factor $H(x)$,

$$H(x)\phi''(x) + H(x)(3-3x)\phi'(x) + \lambda H(x)\phi(x) = 0$$

Then, we see that

$$H(x)\phi''(x) + (3-3x)H(x)\phi'(x) = \frac{d}{dx}\left(H(x)\phi'(x)\right) \text{ if}$$

$$H'(x) = (3-3x)H(x).$$

We can solve this last ODE to find $H(x)$:

$$\int \frac{dH}{H} = \int (3-3x)\,dx \Rightarrow H(x) = C e^{3x - \frac{3}{2}x^2}.$$

We choose $C = 1$ (any $C>0$ would work).

Therefore, the ODE is rewritten in S-L form by

$$\frac{d}{dx}\left(p(x)\,\phi'(x)\right) + g(x)\phi(x) + \lambda\,\sigma(x)\phi(x) = 0 \quad \text{with}$$

$$p(x) = e^{3x-\frac{3}{2}x^2} > 0 \text{ in } [0,2], \quad g(x) = 0,$$

$$\sigma(x) = p(x).$$

b) Verify that $\phi(x) = x(x-1)(x-2)$ is an eigenfunction of the BVP, and compute its eigenvalue.

We need to verify that $\phi(x) = x(x-1)(x-2)$ satisfies the ODE <u>and</u> the BC.

$$\phi(0) = 0 \quad \checkmark \qquad , \quad \phi(x) = (x^2-x)(x-2) = x^3 - 3x^2 + 2x.$$
$$\phi(2) = 0$$

$$\phi'(x) = 3x^2 - 6x + 2$$

$$\phi''(x) = 6x - 6$$

$$\phi''(x) + (3-3x)\phi'(x) + \lambda\phi(x) = 0 \Leftrightarrow$$

$$6x - 6 + (3-3x)(3x^2-6x+2) + \lambda(x^3 - 3x^2 + 2x) = 0 \Leftrightarrow$$

$$x^3(\lambda - 9) + x^2(-3\lambda + 9 + 18) + x(2\lambda + 6 - 18 - 6) + (-6 + 6) = 0 \Leftrightarrow$$

$$\Longleftrightarrow \left| \begin{array}{l} \lambda - 9 = 0, \\ -3\lambda + 9 + 18 = 0, \\ 2\lambda + 6 - 18 - 6 = 0, \\ -6 + 6 = 0. \end{array} \right.$$

All the equations are satisfied for $\lambda = 9$, so $\phi(x) = x(x-1)(x-2)$ is an eigenfunction with eigenvalue $\lambda = 9$.

Note: We have used that a polynomial is identically zero (i.e., zero for all values of x) if all of its coefficients are zero.

c) Prove that the eigenvalue λ obtained in Part b) is the second eigenvalue λ_2 of this Sturm-Liouville problem.

First, notice that this is a regular S-L problem, so all the properties hold.

In particular, property 3] on page -114-:

" Each λ_n has a unique (up to a multiplicative constant) eigenfunction Φ_n with exactly $n-1$ roots in (a,b)".

Here, $(a,b) = (0,2)$. ‖open interval‖

Since $\Phi(x) = x(x-1)(x-2)$ has 1 root in $(0,2)$, given by $x=1$, we deduce that

$n-1 = 1 \to n=2 \to \lambda = 9$ was the second eigenvalue //.