HOMEWORK ASSIGNMENT 8

Name: **Due:** Sunday April 5, 11:59PM (Philadelphia time)

Practice Midterm

Problem 1 [25 points]

Any problem like the first six ones in Homework 6.

Problem 2 [25 points]

Consider the fourth-order eigenvalue problem

$$\frac{d^4\phi}{dx^4} + \lambda\phi = 0, \;\; 0 < x < 1,$$

with boundary conditions given by

$$\phi(0) = 0, \;\; \phi(1) = 0, \;\; \phi'(0) = 0, \;\; \phi''(1) = 0.$$

Use integration by parts when needed in the following questions.

Part a. Show that eigenfunctions corresponding to different eigenvalues are orthogonal.

Part b. Find a *Rayleigh quotient*. That is, given an eigenfunction ϕ, find a formula that determines the corresponding eigenvalue. Use it to show that $\lambda \leq 0$.

Part c. Is $\lambda = 0$ an eigenvalue?

Part d. Assuming the eigenfunctions form a complete set, find the coefficients of the following expansion

$$\sin x = \sum_{\lambda \leq 0} A_\lambda \phi_\lambda(x), \;\; 0 < x < 1.$$

Problem 3 [25 points]

Consider the infinite set of functions $\mathcal{S} = \{\phi_n\}_{n\geq 1}$ with $\phi_n(x) = \sin(2nx)$.

Part a. Show that $\mathcal{S}$ is an orthogonal set on $[0,\pi]$, i.e., that ϕ_n and ϕ_m are orthogonal on $[0,\pi]$ for any $n \neq m$. Find the norm of ϕ_n in $L^2(0,\pi)$.

Part b. Show that the set $\mathcal{S}$ is not complete in $L^2(0,\pi)$.

Part c. Consider the first five eigenfunctions $\{\phi_n\}_{1\leq n\leq 5}$. Find the linear combination of these eigenfunctions that best approximate $f(x) = \cos(2x)$ in $L^2[0,\pi]$ (that is, in the least-square sense). What is the error of this approximation? What happens if we consider 100 eigenfunctions instead of 5?

Part d. Find the numerical inequality one finds after writing and simplifying the Bessel inequality on $\mathcal{S}$ for the function $f(x) = x$ in $L^2[0,\pi]$.

Problem 4 [25 points]

Consider the eigenfunctions that arise from the following eigenvalue problem:

$$\begin{aligned} \phi''(x) + \lambda\phi(x) = 0, \ \ 0 < x < \pi, \\ \phi'(0) = 0, \ \ \phi(\pi) = 0. \end{aligned}$$

Part a. Find the eigenfunctions ϕ_n and compute their L^2 norm.

Part b. Given a function $f(x)$ on $0 \leq x \leq \pi$, find the formula for the coefficients c_n if we assume that

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x), \ \ 0 \leq x \leq \pi.$$

Part c. Let $f \in C^1[0,\pi]$, i.e., both f and f' are continuous functions on $[0,\pi]$. Show that f' belong to $L^2(0,\pi)$.

Part d. Assume that $f \in C^1[0,\pi]$ and that $f'(0) = 0$, $f(\pi) = 0$. Prove the uniform convergence of the series

$$\sum_{n=0}^{\infty} c_n \phi_n(x)$$

to the function $f(x)$ on $(0,\pi)$.

Hint: You might want to find, and use at some point, the eigenfunctions of this other eigenvalue problem

$$\begin{aligned} \phi''(x) + \lambda(x) = 0, \ \ 0 < x < \pi, \\ \phi(0) = 0, \ \ \phi'(\pi) = 0. \end{aligned}$$

Optional but recommended (that is, try them or read their solutions, but no need to write the solutions and submit them)

All the problems below are from W. Strauss book.

1. Section 5.4 (page 134): # 7
2. Section 5.4 (page 135): # 12
3. Section 5.4 (page 135): # 15
4. Section 5.4 (page 136): # 16
5. Section 5.5 (page 145): # 2
6. Section 5.5 (page 145): # 3
7. Section 5.5 (page 145): # 5
8. Section 5.5 (page 145): # 8
9. Section 5.5 (page 145): # 10
10. Section 5.5 (page 145): # 12