HOMEWORK ASSIGNMENT 7, Math 241, Section 002

Name:

Recommended due date:
Friday March 27, 8pm (Philadelphia time).
Deadline:
Tuesday March 31, 8pm (Philadelphia time).

This homework corresponds to Sections 5.3, 5.4, and 7.3 of R. Haberman's book.

1. [**Section 5.3**] Consider the non-Sturm-Liouville differential equation

$$\phi''(x) + \alpha(x)\phi'(x) + (\lambda\beta(x) + \gamma(x))\phi(x) = 0.$$

   Multiply this equation by $H(x)$. Determine $H(x)$ such that the equation may be reduced to the standard Sturm-Liouville form:

$$\frac{d}{dx}\big(p(x)\phi'(x)\big) + q(x)\phi(x) + \lambda\sigma(x)\phi(x) = 0.$$

   Given $\alpha(x), \beta(x)$, and $\gamma(x)$, what are $p(x), \sigma(x)$, and $q(x)$?

2. [**Section 5.3**] Show that $\lambda \geq 0$ for the eigenvalue problem

$$\phi''(x) + (\lambda - x^2)\phi(x) = 0,$$

   with

$$\phi'(0) = 0, \phi'(1) = 0.$$

   Is $\lambda = 0$ an eigenvalue?

3. [**Section 5.3**] Consider the heat flow with convection

$$u_t = ku_{xx} - V_0 u_x.$$

   a) Show that the spatial ordinary differential equation obtained by separation of variables is not in Sturm-Liouville form (V_0 is a constant).
   b) Solve the initial boundary value problem with

$$u_x(0,t) = 0, u_x(L,t) = 0, u(x,0) = f(x)$$

   in terms of the eigenfunctions of the problem $\phi_n(x)$.
   Hint: You may need to transform the spatial ODE into Sturm-Liouville form, check that the problem is a *regular* one, and use some of the properties of these problems.

c) (Optional) In part b), you should have used the Rayleigh coefficient to show that $\lambda \geq 0$. You couldn't prove that $\lambda > 0$. Explain why, indeed, we should expect $\lambda = 0$ to be an eigenvalue (think of an equilibrium solution of this problem).

Remark: In part b), the eigenfunctions and eigenvalues can be computed explicitly, but you are not required to. Notice though that it is very important to know the sign of λ to have a qualitative understanding of the time evolution (and we can do this without explicitly computing the eigenvalues).

4. [**Section 5.4**] Consider

$$c(x)\rho(x)u_t(x,t) = \frac{\partial}{\partial x}\big(K_0(x)u_x(x,t)\big),$$

subject to

$$u_x(0,t) = 0 = u_x(L,t), \;\; u(x,0) = f(x).$$

Assume that the appropriate eigenfunctions are known. Solve the initial value problem, briefly discussing $\lim_{t\to\infty} u(x,t)$.

5. [**Section 7.3**] Solve the Laplace's equation

$$u_{xx} + u_{yy} + u_{zz} = 0,$$

in a rectangular box $0 < x < L$, $0 < y < W$, $0 < z < H$, subject to the boundary conditions

$$u_x(0,y,z) = 0, \;\; u(x,0,z) = 0, \;\; u(x,y,0) = f(x,y),$$
$$u_x(L,y,z) = 0, \;\; u(x,W,z) = 0, \;\; u(x,y,H) = 0.$$

6. Read Sections 7.3, 7.4, and 7.7 of R. Haberman's book.

Recommended further practice problems (Optional):

1. [**Section 7.3**] Solve

$$u_t = k_1 u_{xx} + k_2 u_{yy},$$

on a rectangle $0 < x < L$, $0 < y < H$, subject to

$$u(0,y,t) = 0, \;\; u_y(x,0,t) = 0,$$

$$u(L,y,t) = 0, \;\; u_y(x,H,t) = 0,$$
$$u(x,y,0) = f(x,y).$$

2. **[Section 7.3]** Solve

$$u_{tt} = c^2(u_{xx} + u_{yy}),$$

on a rectangle $0 < x < L$, $0 < y < H$, subject to

$$u_x(0, y, t) = 0, \;\; u_y(x, 0, t) = 0,$$

$$u_x(L, y, t) = 0, \;\; u_y(x, H, t) = 0,$$

$$u(x, y, 0) = 0, u_t(x, y, 0) = f(x, y).$$

3. **[Section 7.3]** Consider the heat equation in a three-dimensional box-shaped region, $0 < x < L$, $0 < y < H$, $0 < z < W$

$$u_t = k(u_{xx} + u_{yy} + u_{zz}),$$

subject to

$$u(0, y, z, t) = 0, \;\; u_y(x, 0, z, t) = 0, \;\; u_z(x, y, 0, t) = 0,$$

$$u(L, y, z, t) = 0, \;\; u_y(x, H, z, t) = 0, \;\; u(x, y, W, t) = 0,$$

$$u(x, y, z, 0) = f(x, y, z).$$

Solve the initial value problem and analyze the temperature as $t \to \infty$.

4. **[Section 5.3]** For the Sturm-Liouville eigenvalue problem,

$$\phi''(x) + \lambda\phi(x) = 0, \quad \phi'(0) = 0 = \phi'(L),$$

verify directly the following general properties:

a) There is an infinite number of eigenvalues with a smallest but no largest.
b) The nth eigenfunction has $n - 1$ zeros.
c) The eigenfunctions are complete and orthogonal.
d) What does the Rayleigh quotient say concerning negative and zero eigenvalues?

5. **[Section 5.3]** Which of the properties of regular Sturm-Liouville problems are valid for the following (non-regular) eigenvalue problem?

$$\phi''(x) + \lambda\phi(x) = 0,$$

$$\phi(-L) = \phi(L), \quad \phi'(-L) = \phi'(L).$$

Remark: The problem is not regular because the boundary conditions are of periodic type.

6. [**Section 5.3**] Consider the eigenvalue problem

$$x^2\phi''(x) + x\phi'(x) + \lambda\phi(x) = 0, \quad \phi(1) = 0, \phi(b) = 0.$$

a) Show that multiplying by $1/x$ transforms this ODE into Sturm-Liouville form.

b) Show that $\lambda \geq 0$.

c) Determine all positive eigenvalues (compute them directly). Is $\lambda = 0$ an eigenvalue? Check that there is an infinite number of eigenvalues with a smallest, but no largest.

d) The eigenfunctions are orthogonal with respect to which weight? Verify this orthogonality using properties of integrals.

e) Show that the nth eigenfunction has $n - 1$ zeros.