Name: ____ **PennID**: __________________

Math 425/AMCS 525
Math 425 - Midterm 1
February 20, 2020, 1:30-2:50pm

Please *turn off and put away all electronic devices*. You are allowed to use one side of a 8x11 cheat-sheet with hand-written notes during this exam. No calculators, no books. Read the problems carefully. **Show all work** (answers without proper justification will not receive full credit). Be as organized as possible: illegible work will not be graded. Please sign and date the pledge below to comply with the Code of Academic Integrity. Don't forget to write your Name and PennID on the top of this page. Good luck!

#	Points possible	Your score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination.

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Signature Date

First-order Linear PDEs: $a(x,y)u_x(x,y) + b(x,y)u_y(x,y) + c(x,y)u(x,y) = f(x,y)$

- Method of Characteristics: Noticing that $au_x + bu_y = (a,b)\cdot\nabla u$, the PDE is transformed into an ODE along an appropriate curve.
  Remark: This curve cannot always be parametrized as $y = y(x)$.

- Method of Coordinates: Look for an adequate change of variables. When a and b are constants, a good change is $\xi = ax + by$, $\eta = bx - ay$.

Second-order Linear PDEs: Heat equation, Wave equation, Laplace equation (and their variations).

- Wave equation:

  1. $u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty \to u(x,t) = f(x+ct) + g(x-ct).$
  2. $u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x) \to$ D'Alembert

$$u(x,t) = \frac{1}{2}\big(\phi(x+ct) - \phi(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds.$$

- Heat equation:

  1. $u_t = ku_{xx}, \quad u(x,0) = H(x), \quad -\infty < x < \infty, \quad (H(x)$ Heaviside function)

$$u(x,y) = \frac{1}{2} + \frac{1}{\sqrt{\pi}}\int_0^{x/\sqrt{4kt}} e^{-s^2} ds.$$

  2. $u_t = ku_{xx}, \quad u(x,0) = \phi(x), \quad -\infty < x < \infty,$

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy, \quad S(x,t) = \frac{1}{\sqrt{4k\pi t}} e^{-\frac{-x^2}{4kt}}.$$

  3. $u_t = ku_{xx} + f(x,t), \quad u(x,0) = \phi(x), \quad -\infty < x < \infty,$

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-\tau)f(y,\tau)dyd\tau.$$

  4. $u_t = ku_{xx}, \quad u(x,0) = \phi(x),\ u(0,t) = 0, \quad 0 < x < \infty,$

$$u(x,t) = \int_0^{\infty} (S(x-y,t) - S(x+y,t))\phi(y)dy.$$

  5. $u_t = ku_{xx}, \quad u(x,0) = \phi(x),\ u_x(0,t) = 0, \quad 0 < x < \infty,$

$$u(x,t) = \int_0^{\infty} (S(x-y,t) + S(x+y,t))\phi(y)dy.$$

Problem 1 (20 pts) Consider the following partial differential equation

$$u_{tt} - 4u_{xx} = -2u_x - u_t, \quad -\infty < x < \infty,\ t > 0. \tag{1}$$

Part a. [7 pts] By choosing an appropriate change of variables (coordinate method), show that the equation can be rewritten in the following form

$$u_{\xi\eta} = -\frac{1}{4}u_\eta, \tag{2}$$

where ξ, η denote the new variables.

Part b. [7 pts] Find the general solution of (2) in terms of η and ξ.

Part c. [6 pts] Assume that the general solution to (1) is given by

$$u(x,t) = f(2t+x)e^{-\frac{1}{4}(2t-x)} + g(2t-x),$$

with f, g any arbitrary smooth functions. Find the solution to (1) together with the initial conditions

$$u(x,0) = -e^{-\frac{x}{4}}, \quad u_t(x,0) = 0.$$

Solution (Problem 1):

Solution (Problem 1):

Problem 2 (20 pts)

Solve $u_x - yu_y + 2u = 1, u(x, 1) = 0$. In what domain in the plane is your solution determined?

Solution (Problem 2):

Solution (Problem 2):

Problem 3 (20 pts): The solution to the heat equation on the real line with initial data the Heaviside function,

$$\begin{aligned} u_t &= ku_{xx}, \quad -\infty < x < \infty, t > 0, \\ u(x,0) &= H(x), \end{aligned}$$

is given in Page 2. Using this fact, prove that the solution to the general inhomogeneous problem

$$\begin{aligned} u_t &= ku_{xx} + f(x,t), \quad -\infty < x < \infty, t > 0, \\ u(x,0) &= \phi(x), \end{aligned}$$

is given by

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-\tau)f(y,\tau)dyd\tau.$$

You may assume that ϕ and f are differentiable functions vanishing at infinity.

Solution (Problem 3):

Solution (Problem 3):

Problem 4 (20 pts) Consider the following wave equation on the real line

$$\begin{aligned} &u_{tt} = 4u_{xx} - \alpha u, \quad -\infty < x < \infty,\ t > 0, \\ &u(x,0) = \phi(x), \\ &u_t(x,0) = \psi(x). \end{aligned}$$

Part a. [5 pts] For $\alpha > 0$, show that if

$$\int_{-\infty}^{\infty} |\phi(x)|^2 dx < \infty, \quad \int_{-\infty}^{\infty} |\phi'(x)|^2 dx < \infty, \quad \int_{-\infty}^{\infty} |\psi(x)|^2 dx < \infty,$$

then

$$\int_{-\infty}^{\infty} |u_t(x,t)|^2 dx < \infty, \quad \int_{-\infty}^{\infty} |u_x(x,t)|^2 dx < \infty.$$

Part b. [5 pts] For $\alpha > 0$ and initial conditions given by

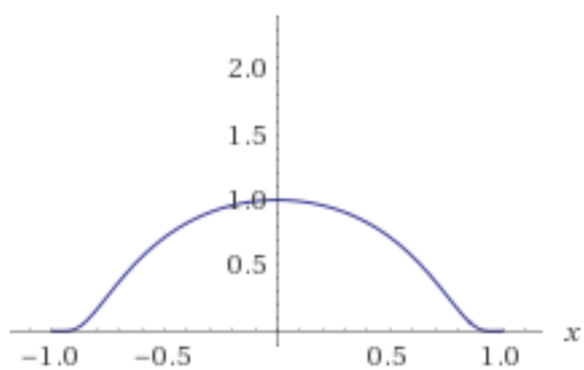
$$\phi(x) = e^{-x^2} \sin x, \quad \psi(x) = \frac{x}{1+x^2},$$

show that the initial value problem has a unique solution.

Part c. [5 pts] In the same conditions as in Part b., show that the solution to the problem is an odd function in x for all time $t \geq 0$. You do not need to find the solution.

Part d. [5 pts] Let $\alpha = 0$. Given the initial conditions below, plot the solution at $t = 1$, $u(x, 1)$. Clearly indicate the maximum value and the support of the solution (i.e., the values of x for which $u(x, 1)$ is not zero).

$$\phi(x) = \begin{cases} e^{-\frac{1}{1-x^2}}, & -1 < x < 1, \\ 0, & \text{otherwise}, \end{cases}, \qquad \psi(x) = 0,$$



Solution (Problem 4):

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Solution (Problem 4):

Problem 5 (20 pts) Solve the following inhomogeneous diffusion problem on the half-line:

$$\begin{aligned}
&u_t - ku_{xx} = f(x,t), \qquad \text{for } 0 < x < \infty, t > 0,\\
&u(0,t) = h(t), \qquad \text{for } 0 < t\\
&u(x,0) = \phi(x), \qquad \text{for } 0 < x < \infty.
\end{aligned}$$

You can use when needed any formula from the cheat-sheet provided on page 2. All other steps must be shown.

Hint: Define a new function $v(x,t)$ in such a way that it satisfies a new problem with an homogeneous boundary condition $v(0,t) = 0$.

Solution (Problem 5):

Solution (Problem 5):

Extra paper

Extra paper

Extra paper