Name: ____________________________

Math 425/
Math 425 -
February 20, 20

Please *turn off and put away all electronic*
a 8x11 cheat-sheet with hand-written notes
Read the problems carefully. **Show all wc**
not receive full credit). Be as organized as possible: illegible work will not be graded
Please sign and date the pledge below to comply with the Code of Academic Integrity.
Don't forget to write your Name and PennID on the top of this page. Good luck!

#	Points possible	Your score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination.

____________________________ ____________________
Signature Date

Problem 1 (20 pts) Consider the following partial differential equation

$$u_{tt} - 4u_{xx} = -2u_x - u_t, \quad -\infty < x < \infty,\ t > 0. \tag{1}$$

Part a. [7 pts] By choosing an appropriate change of variables (coordinate method), show that the equation can be rewritten in the following form

$$u_{\xi\eta} = -\frac{1}{4}u_\eta, \tag{2}$$

where ξ, η denote the new variables.

Part b. [7 pts] Find the general solution of (2) in terms of η and ξ.

Part c. [6 pts] Assume that the general solution to (1) is given by

$$u(x,t) = f(2t+x)e^{-\frac{1}{4}(2t-x)} + g(2t-x),$$

with f, g any arbitrary smooth functions. Find the solution to (1) together with the initial conditions

$$u(x,0) = -e^{-\frac{x}{4}}, \quad u_t(x,0) = 0.$$

Solution (Problem 1):

a) $(\partial_{tt} - 4\partial_{xx}) = (\partial_t - 2\partial_x)(\partial_t + 2\partial_x)$

Let $\eta = 2t - x$, $\xi = 2t + x$ $\rightarrow$ $\partial_x = -\partial_\eta + \partial_\xi$, $\partial_t = 2\partial_\eta + 2\partial_\xi$ $\rightarrow$ $\partial_t - 2\partial_x = 4\partial_\eta$, $\partial_t + 2\partial_x = 4\partial_\xi$

- $16\,\partial_{\eta\xi}u = -2(-u_\eta + u_\xi) - (2u_\eta + 2u_\xi) = -4u_\xi \rightarrow$

$\rightarrow u_{\eta\xi} = \frac{-1}{4}u_\xi$ (interchange ξ, η)

b) Let $v(\eta,\xi) = u_\eta(\eta,\xi) \rightarrow v_\xi = -\frac{1}{4}v \Rightarrow v = c(\eta)e^{\frac{-1}{4}\xi} \rightarrow$

$\rightarrow u(\xi,\eta) = f(\eta)e^{\frac{-1}{4}\xi} + g(\xi)$

Solution (Problem 1):

c)
$$\begin{cases} -e^{-\frac{x}{4}} = f(x)\,e^{\frac{x}{4}} + g(-x) \\ 0 = 2f'(x)\,e^{\frac{x}{4}} - \frac{1}{2} f(x)\,e^{x/4} + 2g'(-x) \end{cases}$$

$$\frac{1}{4} e^{-\frac{x}{4}} = f'(x)\,e^{\frac{x}{4}} + \frac{1}{4} f(x)\,e^{x/4} - g'(-x) \Rightarrow$$

$$\begin{cases} \frac{1}{2} e^{-\frac{x}{4}} = 2f'(x)\,e^{x/4} + \frac{1}{2} f(x)\,e^{x/4} - 2g'(-x) \\ 0 = 2f'(x)\,e^{x/4} - \frac{1}{2} f(x)\,e^{x/4} + 2g'(-x) \end{cases}$$

$$\frac{1}{2} e^{-\frac{x}{4}} = 4f'(x)\,e^{x/4} \Rightarrow f'(x) = \frac{1}{8} e^{-x/2} \Rightarrow f(x) = -\frac{1}{4} e^{-x/2} + c.$$

$$g(-x) = -e^{-x/4} - e^{x/4}\left(-\frac{1}{4} e^{-x/2} + c\right) \Rightarrow$$

$$\Rightarrow g(x) = -e^{x/4} - e^{-x/4}\left(-\frac{1}{4} e^{x/2} + c\right) = -e^{x/4} + \frac{1}{4} e^{x/4} - c\,e^{-x/4}.$$

$$u(x,t) = \left(-\frac{1}{4} e^{-\frac{2t+x}{2}} + \cancel{c}\right) e^{-\frac{1}{4}(2t-x)} + -e^{\frac{2t-x}{4}} + \frac{1}{4} e^{\frac{2t-x}{4}} - \cancel{c\,e^{-\frac{2t+x}{4}}} =$$

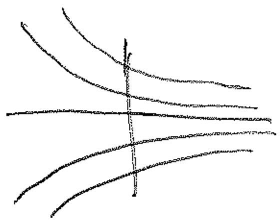
$$= -\frac{1}{4} e^{-\frac{3}{2}t - \frac{x}{4}} - \frac{3}{4} e^{\frac{2t-x}{4}}.$$

Problem 2 (20 pts)

Solve $u_x - yu_y + 2u = 1, u(x,1) = 0$. In what domain in the plane is your solution determined?

Solution (Problem 2):

$$y = y(x) \to \frac{dy}{dx} = -y \to \ln|y| = -x + c \to |y| = c e^{-x}$$



Characteristic curves.

$$v(x) = u(x, y(x)) \to \frac{dv}{dx} = u_x(x, y(x)) + u_y(x, y(x))\, y'(x) =$$

$$= u_x(x, y(x)) - y(x)\, u_y(x, y(x))$$

Thus,

$$v'(x) + 2v = 1$$

$$v(x_0) = u(x_0, y(x_0)) = u(x_0, c e^{-x_0})$$

Choose $x_0 = \ln c \to v(\ln c) = u(\ln c, 1) = 0$ $\quad$ $\left(\begin{matrix}\text{Here,} \\ c > 0\end{matrix}\right).$

$$\bullet \left.\begin{matrix} v'(x) + 2v = 1 \\ v(\ln c) = 0 \end{matrix}\right\} \to v(x) = \frac{1}{2} + c_1 e^{-2x}$$

$$v(\ln c) = \frac{1}{2} + c_1 e^{-2\ln c} = 0 \Leftrightarrow c_1 e^{\ln \frac{1}{c^2}} = \frac{-1}{2} \Leftrightarrow$$

$$\Leftrightarrow c_1 \frac{1}{c^2} = \frac{-1}{2} \Leftrightarrow c_1 = -\frac{c^2}{2}.$$

Solution (Problem 2):

- So, $v(x) = \frac{1}{2} - \frac{c^2}{2} e^{-2x}$.

- Back to x, y: $\qquad c = ye^{x}$

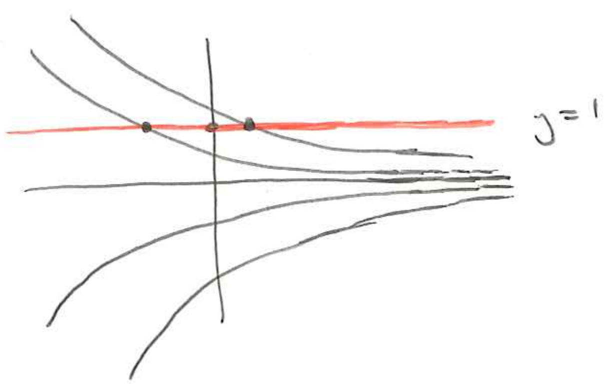
$$u(x, y(x)) = u(x, ce^{-x}) = \frac{1}{2} - \frac{c^2}{2} e^{-2x} \Rightarrow$$

$$\Rightarrow u(x,y) = \frac{1}{2} - \frac{1}{2} y^2 \,//$$

- Region: $c > 0 \rightsquigarrow ye^{x} > 0 \rightarrow y > 0$. That is,

$\{(x,y) \in \mathbb{R}^2 : y > 0\}$.

Also, from the picture



Problem 3 (20 pts): The solution to the heat equation on the real line with initial data the Heaviside function,

$$u_t = ku_{xx}, \quad -\infty < x < \infty, t > 0,$$
$$u(x, 0) = H(x),$$

is given in Page 2. Using this fact, prove that the solution to the general inhomogeneous problem

$$u_t = ku_{xx} + f(x, t), \quad -\infty < x < \infty, t > 0,$$
$$u(x, 0) = \phi(x),$$

is given by

$$u(x, t) = \int_{-\infty}^{\infty} S(x - y, t)\phi(y)dy + \int_0^t \int_{-\infty}^{\infty} S(x - y, t - \tau)f(y, \tau)dyd\tau.$$

You may assume that ϕ and f are differentiable functions vanishing at infinity.

PDE

Solution (Problem 3):

$$u_t(x,t) = \int_{-\infty}^{\infty} S_t(x-y,t)\phi(y)\,dy + \int_0^t \int_{-\infty}^{\infty} S_t(x-y,t-\tau) f(y,\tau)\,dy\,d\tau +$$

$$+ \lim_{\tau \to t^-} \int_{-\infty}^{\infty} S(x-y,t-\tau) f(y,\tau)\,dy$$

FS

$$u_{xx}(x,t) = \int_{-\infty}^{\infty} S_{xx}(x-y,t)\phi(y)\,dy + \int_0^t \int_{-\infty}^{\infty} S_{xx}(x-y,t-\tau) f(y,\tau)\,dy\,d\tau.$$

Thus,

$$u_t(x,t) - ku_{xx} = \int_{-\infty}^{\infty} \underbrace{(S_t(x-y,t) - kS_{xx}(x-y,t))}_{=0}\phi(y)\,dy +$$

$$+ \int_0^t \int_{-\infty}^{\infty} \underbrace{(S_t(x-y,t-\tau) - kS_{xx}(x-y,t-\tau))}_{=0} f(y,\tau)\,dy\,d\tau + \lim_{\tau \to t} \int_{-\infty}^{\infty} Q_x(x-y,t-\tau) f(y,\tau)\,dy.$$

Solution (Problem 3):

where we have use that $S = Q_x$ by definition, and therefore it solves the homogeneous heat equation (Q solves it, so any derivative of Q is also a solution).

- We finally show that the last limit equals $f(x,t)$:

$$\lim_{\tau\to t^-}\int_{-\infty}^{\infty} Q_x(x-y,t-\tau)f(y,\tau)dy = \lim_{\tau\to t^-}\int_{-\infty}^{\infty} -Q_y(x-y,t-\tau)f(y,\tau)dy =$$

$$= \lim_{\tau\to t^-}\int_{-\infty}^{\infty} Q(x-y,t-\tau)\partial_y f(y,\tau)dy + \lim_{\tau\to t^-}\Big[-Q(x-y,t-\tau)\cancel{f(y,\tau)}\Big]_{-\infty}^{\infty} =$$

$$= \int_{-\infty}^{\infty} \underbrace{Q(x-y,0)}_{=H(x-y)}\partial_y f(y,t)dy = \int_{-\infty}^{x} \partial_y f(y,t)dy = f(x,t) - \cancel{f(y,t)}\Big|_{y=-\infty} = f(x,t).$$

$\boxed{IC}$ $\boxed{5}$

$$u(x,0) = \lim_{t\to 0^+}\int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy + 0 = \lim_{t\to 0^+}\int_{-\infty}^{\infty} Q_x(x-y,t)\phi(y)dy =$$

$$= \lim_{t\to 0^+}\int_{-\infty}^{\infty} Q(x-y,t)\phi'(y)dy + \lim_{t\to 0^+}\Big[-Q(x-y,t)\cancel{\phi(y)}\Big]_{-\infty}^{\infty} =$$

$$= \int_{-\infty}^{\infty} \underbrace{Q(x-y,0)}_{=H(x-y)}\phi'(y)dy = \int_{-\infty}^{x}\phi'(y)dy = \phi(x) - \cancel{\phi(y)}\Big|_{y=-\infty} = \phi(x).$$

Problem 4 (20 pts) Consider the following wave equation on the real line

$$
\begin{aligned}
&u_{tt} = 4u_{xx} - \alpha u, \quad -\infty < x < \infty,\ t > 0,\\
&u(x,0) = \phi(x),\\
&u_t(x,0) = \psi(x).
\end{aligned}
$$

Part a. [5 pts] For $\alpha > 0$, show that if

$$\int_{-\infty}^{\infty} |\phi(x)|^2 dx < \infty, \quad \int_{-\infty}^{\infty} |\phi'(x)|^2 dx < \infty, \quad \int_{-\infty}^{\infty} |\psi(x)|^2 dx < \infty,$$

then

$$\int_{-\infty}^{\infty} |u_t(x,t)|^2 dx < \infty, \quad \int_{-\infty}^{\infty} |u_x(x,t)|^2 dx < \infty.$$

Part b. [5 pts] For $\alpha > 0$ and initial conditions given by

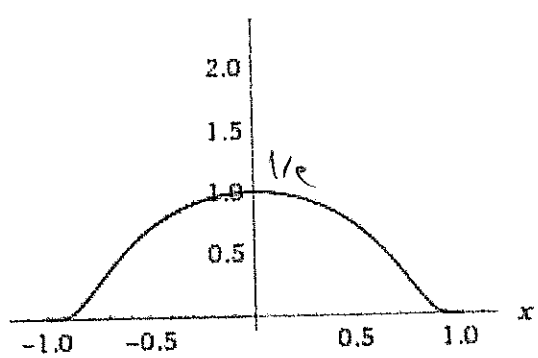
$$\phi(x) = e^{-x^2} \sin x, \quad \psi(x) = \frac{x}{1+x^2},$$

show that the initial value problem has a unique solution.

Part c. [5 pts] In the same conditions as in Part b., show that the solution to the problem is an odd function in x for all time $t \geq 0$. You do not need to find the solution.

Part d. [5 pts] Let $\alpha = 0$. Given the initial conditions below, plot the solution at $t = 1$, $u(x,1)$. Clearly indicate the maximum value and the support of the solution (i.e., the values of x for which $u(x,1)$ is not zero).

$$\phi(x) = \begin{cases} e^{-\frac{1}{1-x^2}}, & -1 < x < 1, \\ 0, & \text{otherwise}, \end{cases} \quad \psi(x) = 0,$$



Solution (Problem 4):

Solution (Problem 4):

a) $u_{tt} u_t = 4 u_{xx} u_t - \alpha u u_t \to$ + 1

$$\to \int_{-\infty}^{\infty} \frac{1}{2} \frac{\partial}{\partial t} |u_t|^2 \, dx = 4 \int_{-\infty}^{\infty} u_{xx} u_t \, dx - \alpha \int_{-\infty}^{\infty} \frac{1}{2} \frac{\partial}{\partial t} |u|^2 \, dx \Rightarrow$$

+ 1

$$\Rightarrow \frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} |u_t|^2 \, dx = -4 \int_{-\infty}^{\infty} \underbrace{u_x u_{xt}}_{\frac{1}{2}\frac{\partial}{\partial t}|u_x|^2} \, dx + 4 \cancel{u_x u_t \Big]_{-\infty}^{\infty}} - \frac{\alpha}{2} \frac{d}{dt} \int_{-\infty}^{\infty} |u|^2 dx \Rightarrow$$

+ 1

$$\Rightarrow \frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} \left(|u_t|^2 + 4 |u_x|^2 + \alpha |u|^2 \right) dx = 0 \Rightarrow$$

+ 1

$$\int_{-\infty}^{\infty} \left(|u_t|^2 + 4 |u_x|^2 + \alpha |u|^2 \right) dx = \int_{-\infty}^{\infty} \left(|\psi(x)|^2 + 4 |\phi'(x)|^2 + \alpha |\phi(x)|^2 \right) dx < \infty \, //.$$

b) By contradiction: Assume u_1, u_2 are two different solutions.

Let $v = u_1 - u_2 \neq 0$. Then, v satisfies
$$\left. \begin{aligned} & v_{tt} = 4 v_{xx} - \alpha v, \\ & v(x,0) = 0, \\ & v_t(x,0) = 0, \end{aligned} \right\}$$

+ 1 + 1

Using part a), this implies that

+ 3

$$\int_{-\infty}^{\infty} \left(|v_t|^2 + 4 |v_x|^2 + \alpha |v|^2 \right) dx = 0 \Rightarrow v \equiv 0 \Rightarrow u_1 = u_2 \; \not\leftarrow .$$

Solution (Problem 4):

+ 2

c) Let $v(x,t) = u(x,t) + u(-x,t)$.

Then, v satisfies:

$$v_{tt}(x,t) = u_{tt}(x,t) + u_{tt}(-x,t),$$
$$v_{xx}(x,t) = u_{xx}(x,t) + u_{xx}(-x,t)$$

$$v_{tt}(x,t) = 4u_{xx}(x,t) + 4u_{xx}(-x,t) - \alpha u(x,t) - \alpha u(-x,t) =$$
$$= 4v_{xx}(x,t) - \alpha v(x,t),$$

+ 1.5

with $v(x,0) = \phi(x) + \phi(-x) = 0$, because $\phi(x)$ is odd.

$v_t(x,0) = \psi(x) + \psi(-x) = 0$, because $\psi(x)$ is odd.

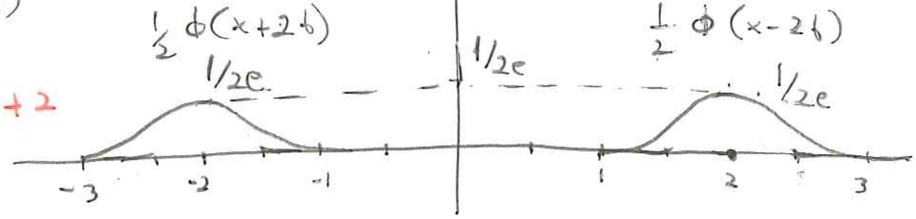
+ 1.5

By part b), this implies that $v \equiv 0$, and thus $u(x,t) = -u(-x,t)$. That is, $u(x,t)$ is odd in x for all $t \geq 0$.

d) D'Alembert: $u(x,t) = \frac{1}{2}(\phi(x+2t) + \phi(x-2t))$. + 3

$\underline{\underline{t=1}}$ (traslation by $\pm$.)

+ 2



Problem 5 (20 pts) Solve the following inhomogeneous diffusion problem on the half-line:

$$u_t - ku_{xx} = f(x,t), \qquad \text{for } 0 < x < \infty, t > 0,$$
$$u(0,t) = h(t), \qquad \text{for } 0 < t$$
$$u(x,0) = \phi(x), \qquad \text{for } 0 < x < \infty.$$

You can use when needed any formula from the cheat-sheet provided on page 2. All other steps must be shown.

Hint: Define a new function $v(x,t)$ in such a way that it satisfies a new problem with an homogeneous boundary condition $v(0,t) = 0$.

Solution (Problem 5):

See notes.

- $v(x,t) = u(x,t) - h(t)$. + 5
- Write problem for v: $\left.\begin{cases} v_t - kv_{xx} = f(x,t) - h'(t) \\ v(0,t) = 0 \\ v(x,0) = \phi(x) - h(0) \end{cases}\right\}$ + 5
- Odd extension: $\begin{cases} w_t - kw_{xx} = F_{odd}(x,t), \quad -\infty < x < \infty \\ w(x,0) = \varphi_{odd}(x) \end{cases}$ + 5

$F_{odd}(x,t) = \begin{cases} f(x,t) - h'(t), & x > 0 \\ 0 & x = 0 \\ -f(-x,t) + h'(t), & x < 0 \end{cases}$

$\varphi_{odd}(x) = \begin{cases} \phi(x) - h(0), & x > 0 \\ 0 & x = 0 \\ -\phi(-x) + h(0), & x < 0 \end{cases}$

- Solve for w
+ 5
- $v = w\big|_{x>0}$

Solution (Problem 5):

Extra paper

Extra paper

Extra paper