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Math 425/AMCS 525
Math 425 - Midterm 1 - Practice Exam
February 20, 2020, 1:30-2:50pm

Please *turn off and put away all electronic devices.* You are allowed to use one side of a 8x11 cheat-sheet with hand-written notes during this exam. No calculators, no books. Read the problems carefully. **Show all work** (answers without proper justification will not receive full credit). Be as organized as possible: illegible work will not be graded. Please sign and date the pledge below to comply with the Code of Academic Integrity. Don't forget to write your Name and PennID on the top of this page. Good luck!

#	Points possible	Your score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination.

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Signature Date

Problem 1 (20 pts)

Solve

$$3u_{xx} + 10u_{xt} + 3u_{tt} = 0,$$
$$u(x,0) = e^{-x}\sin x, \quad u_t(x,0) = \frac{1}{1+x^2}.$$

Solution (Problem 1):

Factor the operator: $3u_{xx} + 10u_{xt} + 3u_{tt} = (3\partial_t + \partial_x)(\partial_t + 3\partial_x)u$

The general solution is

$u(x,t) = f(t-3x) + g(3t-x)$.

[Coordinate method; transport eq. ...]

I.C:

$$\left.\begin{aligned} e^{-x}\sin x &= f(-3x) + g(-x), \\ \frac{1}{1+x^2} &= f'(-3x) + 3g'(-x) \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{aligned} -e^{-x}\sin x + e^{-x}\cos x &= -3f'(-3x) - g'(-x) \\ \frac{1}{1+x^2} &= f'(-3x) + 3g'(-x) \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow 8g'(-x) = -e^{-x}\sin x + e^{-x}\cos x + \frac{3}{1+x^2} \Rightarrow$$

$$\Rightarrow g'(x) = \frac{1}{8}e^{x}\sin x + \frac{1}{8}e^{x}\cos x + \frac{3/8}{1+x^2} \Rightarrow \left\{ \begin{aligned} &e^{x}\sin x + e^{x}\cos x = \\ &= \frac{\partial}{\partial x}(e^{x}\sin x) \end{aligned} \right\},$$

Solution (Problem 1):

$$g(x) = \frac{1}{8} e^{x} \sin x + \frac{3}{8} \operatorname{arctg}(x) + c$$

$$f(-3x) = e^{-x} \sin x - g(-x) = e^{-x} \sin x + \frac{1}{8} e^{-x} \sin x +$$

$$+ \frac{3}{8} \operatorname{arctg}(x) - c \Rightarrow$$

$$\Rightarrow f(x) = -e^{x/3} \sin\left(\frac{x}{3}\right) - \frac{1}{8} e^{x/3} \sin\left(\frac{x}{3}\right) - \frac{3}{8} \operatorname{arctg}\left(\frac{x}{3}\right) - c$$

In summary,

$$u(x,t) = f(t-3x) + g(3t-x) =$$

$$= -e^{\frac{t}{3}-x} \sin\left(\frac{t}{3}-x\right) - \frac{1}{8} e^{\frac{t}{3}-x} \sin\left(\frac{t}{3}-x\right) - \frac{3}{8} \operatorname{arctg}\left(\frac{t}{3}-x\right) +$$

$$+ \frac{1}{8} e^{3t-x} \sin(3t-x) + \frac{3}{8} \operatorname{arctg}(3t-x).$$

Problem 2 (20 pts)

Use the method of characteristics to solve

$$\sin(y)u_x + 2u_y + u = 1,$$
$$u(0, y) = \cos(y),$$

and find the region of the xy plane in which the solution is uniquely determined.

Solution (Problem 2):

→ Problem 3 of last year midterm (file on Canvas "Midterm 1 - Solutions").

Solution (Problem 2):

Problem 3 (20 pts): Give a proof of the weak maximum principle for the Laplace equation on a rectangle:

$$u_{xx} + u_{yy} = 0, \quad 0 \le x \le a, \quad 0 \le y \le b.$$

The weak maximum principle here says: if $u(x, y)$ is a solution, then the maximum of $u(x, y)$ on the whole rectangle $0 \le x \le a$, $0 \le y \le b$ is equal to the maximum of $u(x, y)$ on the boundaries (i.e., on $x = 0$ or $x = a$ or $y = 0$ or $y = b$).

Hint: Proceed as in the heat equation, by defining an auxiliary function $v(x, y) = u(x, y) + \varepsilon|x|^2 = u(x, y) + \varepsilon(x^2 + y^2)$ ($\varepsilon > 0$).

Solution (Problem 3):

Problem 6 last year midterm. Solution on Canvas.

Solution (Problem 3):

Problem 4 (20 pts) Consider the following heat equation on the real line

$$u_t = 3u_{xx} - u, \quad -\infty < x < \infty,\ t > 0,$$
$$u(x,0) = e^{-|x|}.$$

Part a. Prove that the initial value problem has a unique solution.

Part b. Show that the solution to the problem is an even function in x for all time $t \geq 0$. You do not need to find the solution.

Solution (Problem 4):

a) Proof by contradiction. Assume $u_1 \neq u_2$ are two solutions. Define $v = u_1 - u_2$. Then,

$$\begin{cases} v_t = 3v_{xx} - v \\ v(x,0) = 0 \end{cases}$$

Multiplying the equation by v and integrating,

$$\int_{-\infty}^{\infty} v_t(x,t)\, v(x,t)\, dx = 3\int_{-\infty}^{\infty} v_{xx}(x,t)\, v(x,t)\, dx - \int_{-\infty}^{\infty} (v(x,t))^2\, dx \Rightarrow$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} |v(x,t)|^2\, dx = -3\int_{-\infty}^{\infty} |v_x(x,t)|^2\, dx + \cancel{3\, v_x(x,t)\, v(x,t)\Big]_{-\infty}^{\infty}}_{(*)} +$$

$$-3\int_{-\infty}^{\infty} |v(x,t)|^2\, dx \Rightarrow$$

$$\Rightarrow \int_{-\infty}^{\infty} |v(x,t)|^2\, dx \leq \int_{-\infty}^{\infty} (v(x,0))^2\, dx = 0 \Rightarrow v \equiv 0 \Rightarrow u_1 = u_2 \Rightarrow \text{↯}.$$

(* extra page at the end.)

Problem 5 (20 pts) Solve the following inhomogeneous diffusion problem on the half-line:

$$
\begin{aligned}
&u_t - ku_{xx} = f(x,t), \qquad && \text{for } 0 < x < \infty, t > 0,\\
&u_x(0,t) = h(t), && \text{for } 0 < t\\
&u(x,0) = \phi(x), && \text{for } 0 < x < \infty.
\end{aligned}
$$

You can use when needed any formula from the cheat-sheet provided on page 2. All other steps must be shown.

Hint: Define a new function $v(x,t)$ in such a way that it satisfies a new problem with an homogeneous boundary condition $v_x(0,t) = 0$.

Solution (Problem 5):

Let $v(x,t) = u(x,t) - x h(t)$. Then,

$v_t(x,t) = u_t(x,t) - x h'(t)$,

$v_{xx}(x,t) = u_{xx}(x,t)$,

$$
\left.\begin{aligned}
&v_t - k v_{xx} = -x h'(t) + f(x,t)\\
&v_x(0,t) = 0\\
&v(x,0) = \phi(x) - x \cdot h(0)
\end{aligned}\right\}
$$

Let $w(x,t)$ solve

$w_t - k w_{xx} = F_{even}$, $-\infty < x < \infty$,

$w(x,0) = G_{even}$,

where $F_{even}(x) = \begin{cases} -x h'(t) + f(x,t), & x \geq 0. \\ x h'(t) + f(-x,t), & x < 0 \end{cases}$.

Solution (Problem 5):

$$G_{even}(x) = \begin{cases} \phi(x) - x h(0), & x \geq 0 \\ \phi(-x) + x h(0), & x < 0 \end{cases}.$$

Then,

$$w(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\, G_{even}(y)\,dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-\tau)\, F_{even}(y,\tau)\,dy\,d\tau =$$

$$= \int_0^{\infty} \Big(S(x-y,t) + S(x+y,t)\Big)\Big(\phi(y) - y h(0)\Big)\,dy +$$

$$+ \int_0^t \int_0^{\infty} \Big(S(x-y,t-\tau) + S(x+y,t-\tau)\Big)\Big(-y h'(\tau) + f(y,\tau)\Big)\,dy\,d\tau$$

• Going back,

$$v(x,t) = w(x,t)\Big|_{x>0} \qquad \left(\begin{array}{c} v_t - k v_{xx} = w_t - k w_{xx} = F_{even}(x,t) = \\ = F(x,t) \text{ for } x>0, \\ v(x,0) = w(x,0) = G_{even}(x) = G(x) \text{ for } x>0 \end{array} \right).$$

and

$$u(x,t) = v(x,t) + x h(t) \,/\!/ .$$

Extra paper

(*) we are proving uniqueness in the class of solutions that vanish at $\pm\infty$).

b) Let $v(x,t) = u(x,t) - u(-x,t)$.

Then, $v_t(x,t) = u_t(x,t) - u_t(-x,t)$,

$v_{xx}(x,t) = u_{xx}(x,t) - u_{xx}(-x,t)$,

so

$$\left.\begin{array}{l} v_t = 3v_{xx} - v \\ v(x,0) = u(x,0) - u(-x,0) = e^{-|x|} - e^{-|x|} = 0 \end{array}\right\} \Rightarrow$$

$\Rightarrow v(x,t) = 0$ for all $x \in \mathbb{R}$, $t \geq 0$ (part a.)

Therefore, $u(x,t) = u(-x,t)$ for all $x \in \mathbb{R}$, $t \geq 0$ //.

Extra paper

Extra paper