Name ______________________________________________

**Math 241 - Section 002 - Midterm 1**
**Thursday, February 20, 2020, @ 10:30 AM - 11:50 AM**

No form of cheating will be tolerated. You are expected to uphold the Code of Academic Integrity. I certify that all of the work on this test is my own.

**Signature:** ______________________________

The use of computers, smartphones, and similar devices is neither necessary nor permitted during this exam. Correct answers without proper justification will not receive full credit. Clearly highlight your answers and the steps taken to arrive at them: illegible work will not be graded. No cheat-sheet is allowed.

**OFFICIAL USE ONLY:**

| Problem | Points | Your score |
|---|---|---|
| 1 | 20 | |
| 2 | 20 | |
| 3 | 20 | |
| 4 | 20 | |
| 5 | 20 | |
| Total | 100 | |

**Problem 1** [20 points]

This problem deals with the diffusion of heat along a straight rod of length $L$, with constant sectional area. Assume that the rod has constant specific heat $c$, constant density $\rho$, and a constant heat conductivity $K$.

**Part a.** [4 points] There is electricity going through the rod, so that it produces a generation of heat per volume and time $Q(x,t)$. State in mathematical terms the conservation of energy in an arbitrary region of the rod between $x = a$ and $x = b$.

(see book)

**Part b.** [4 points] Denote by $u(x,t)$ the temperature at each point $x$ and time instant $t$. Assuming that Fourier's law of diffusion is satisfied,

$$\phi = -K\frac{\partial u}{\partial x},$$

show that the partial differential equation that models the evolution of $u(x,t)$ is

$$u_t = ku_{xx} + Q/(c\rho),$$

where $k = K/(c\rho)$.

(see book)

$x=0$, $x=L$

**Part c.** [6 points] If $k = 1$, $Q(x,t)/(c\rho) = x - \beta$, and both ends of the rod, $x = a$, $x = b$, are insulated, determine how $\beta$ must be related to $L$ if a time-independent (equilibrium) temperature distribution is to exist. Explain in physical terms the meaning of $\beta$.

$$u_t = u_{xx} + x - \beta \qquad u_{xx} = \beta - x \rightarrow u_x = \beta x - x^2/2 + c_1$$

$$u_x(0,t) = 0 = u_x(L,t)$$

$$\hookrightarrow u_x(0) = c_1 = 0$$

$$u_x(L) = L\beta - \frac{L^2}{2} = 0 \rightarrow \boxed{\beta = \frac{L}{2}}$$

Since the bounds of the rod are insulated, no heat enters/leaves the rod. In order for an equilibrium sol. to exist, the total $\Delta$ in heat energy must be constant. The $\frac{Q}{c\rho}$ term $x - \beta$ must have a value for $\beta$ that makes the total heat generated in the rod 0

**Part d.** [6 points] Find the equilibrium solution if $u(x,0) = f(x)$.

$$u(x) = \frac{L}{2}\frac{x^2}{2} - \frac{x^3}{6} + c_2$$

$$u_t = u_{xx} + x - \frac{L}{2} \rightarrow \frac{d}{dt}\int_0^L u(x,t)\,dx = \left[\frac{x^2}{2} - \frac{L}{2}x\right]_0^L = 0 \Rightarrow$$

$$\Rightarrow \int_0^L u(x,t)\,dx = \int_0^L f(x)\,dx \Rightarrow$$

$$\Rightarrow \left[\frac{L}{2}\frac{x^3}{6} - \frac{x^4}{24} + c_2 x\right]_0^L = \frac{L^4}{12} - \frac{L^4}{24} + c_2 L = \frac{L^4}{24} + c_2 L = \int_0^L f(x)\,dx \Rightarrow$$

$$\Rightarrow c_2 = \frac{1}{L}\int_0^L f(x)\,dx - \frac{L^3}{24}$$

**Problem 2** [20 points]

Consider the heat equation

**Part a.** [8 points] Find the eigenvalues and eigenfunctions for the following boundary value problem:

$$\phi''(x) + \lambda\phi(x) = 0, \quad 0 < x < 1$$
$$\phi'(0) = 0, \quad \phi(1) = 0.$$

Consider all possible values of $\lambda$. Show all the steps.

$\underline{\lambda = \alpha^2 > 0}$

$\phi''(x) + \alpha^2 \phi(x) = 0 \Rightarrow \phi(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$

$\phi'(x) = -c_1 \alpha \sin(\alpha x) + c_2 \alpha \cos(\alpha x)$

$\phi'(0) = c_2 \alpha = 0 \Rightarrow c_2 = 0$

$\phi(1) = c_1 \cos(\alpha) = 0 \Leftrightarrow \alpha = (2k-1)\frac{\pi}{2}, \; k = 1, 2, \ldots$

$\lambda = \left((2k-1)\frac{\pi}{2}\right)^2, \; k = 1 \ldots$

$\phi(x) = \cos(\sqrt{\lambda}\, x)$

$\underline{\lambda = 0}$ : $\phi(x) = c_1 x + c_2$

$\phi'(x) = c_1 \Rightarrow \phi'(0) = c_1 = 0$

$\phi(1) = c_2 = 0$

$\Rightarrow \lambda = 0$ is not a eigval.

$\underline{\lambda = -\alpha^2 < 0}$ : $\phi(x) = c_1 \cosh(x) + c_2 \sinh(x)$

$\phi'(x) = c_1 \sinh(x) + c_2 \cosh(x)$

$\phi'(0) = c_2 = 0$

$\phi(1) = c_1 \cosh(1) = 0 \Rightarrow c_1 = 0$

$\Rightarrow$ No negative eigenvalue.

**Part b.** [12 points] Solve the 1D heat equation $u_t = 2u_{xx}$ for $0 < x < 1$, $t > 0$ subject to $u_x(0,t) = 0$, $u(1,t) = 0$, and initial condition

$$u(x,0) = 2\cos\left(\frac{\pi}{2}x\right) + 3\cos\left(\frac{5\pi}{2}x\right).$$

Hint: Use Part a.

$$u(x,t) = F(x)\,G(t)$$

$$G'(t)F(x) = 2F''(x)G(t) \rightsquigarrow \frac{1}{2}\frac{G'(t)}{G(t)} = \frac{F''(x)}{F(x)} = -\lambda \rightsquigarrow$$

$$\rightsquigarrow \text{①}\ F''(x) + \lambda F(x) = 0, \qquad \text{②}\ G'(t) = -2\lambda G(t).$$

$\underline{BC}$: $$\left.\begin{aligned} &u_x(0,t) = 0 = F'(0)G(t) \rightsquigarrow F'(0) = 0 \\ &u(1,t) = 0 = F(1)G(t) \Rightarrow F(1) = 0 \end{aligned}\right\}$$

①
$$\left.\begin{aligned} &F''(x) + \lambda F(x) = 0 \\ &F'(0) = 0 = F(1) \end{aligned}\right\} \xrightarrow{a)} \lambda = \left((2k-1)\frac{\pi}{2}\right)^2, \quad k = 1, 2, \ldots$$

$$F(x) = \cos\left((2k-1)\frac{\pi}{2}x\right)$$

②
$$G'(t) = -2(2k-1)\frac{\pi}{2}G(t) \rightsquigarrow G(t) = e^{-2\left((2k-1)\frac{\pi}{2}\right)^2 t}.$$

- $$u(x,t) = \sum_{k=1}^{\infty} C_k \cos\left((2k-1)\frac{\pi}{2}x\right) e^{-2\left((2k-1)\frac{\pi}{2}\right)^2 t}.$$

- From $u(x,0)$, we obtain that $C_1 = 2$, $C_3 = 3$,

$C_k = 0$ for $k \neq 1, 3$

$$\therefore \boxed{u(x,t) = 2\cos\left(\frac{\pi x}{2}\right) e^{-2\left(\frac{\pi}{2}\right)^2 t} + 3\cos 5\left(\frac{5\pi x}{2}\right) e^{-2\left(\frac{5\pi}{2}\right)^2 t}}$$

$\cancel{\Rightarrow}$ $F(0)=1$ ! This is false

**Problem 3** [20 points]

**Part a.** [12 points] Solve the Laplace equation on a rectangle

$$u_{xx}+u_{yy}=0, \qquad 0<x<a,\ 0<y<b,$$

with the following boundary conditions: $\rightarrow$ Non-Homogeneous BC $\rightarrow$ Leave until the end! (after superposition principle).

$$\boxed{u(0,y)=1,} \quad u(a,y)=0, \quad 0<y<b,$$

and

$$u_y(x,0), \quad u_y(x,b)=0, \quad 0<x<a.$$

$$u(x,y)=F(x)G(y) \rightarrow \frac{F''(x)}{F(x)}+\frac{G''(y)}{G(y)}=0 \Rightarrow \frac{F''(x)}{F(x)}=-\frac{G''(y)}{G(y)}=\lambda \quad (+1)$$

+4/4

BC: $F(a)=0$

$G'(0)=G'(b)=0$

$$\left.\begin{array}{l} G''(y)+\lambda G(y)=0 \\ G'(0)=G'(b)=0 \end{array}\right\} \Rightarrow \lambda=\left(\frac{n\pi}{b}\right)^2,\ n=0,1,\ldots$$

(+1)

$$G(y)=\cos\left(\frac{n\pi y}{b}\right) \quad (+1)$$

- $F''(x)-\left(\frac{n\pi}{b}\right)^2F(x)=0$ (+1)

$\boxed{n\neq0}$ $\hookrightarrow F(x)=c_1\cosh\left(\frac{n\pi x}{b}\right)+c_2\sinh\left(\frac{n\pi x}{b}\right)$

$$F(a)=0=c_1\cosh\left(\frac{n\pi a}{b}\right)+c_2\sinh\left(\frac{n\pi a}{b}\right) \Rightarrow c_2=-c_1\frac{\cosh\left(\frac{n\pi a}{b}\right)}{\sinh\left(\frac{n\pi a}{b}\right)},$$

+2/2

$$F(x)=c_1\cosh\left(\frac{n\pi x}{b}\right)-c_1\frac{\cosh\left(\frac{n\pi a}{b}\right)}{\sinh\left(\frac{n\pi a}{b}\right)}\sinh\left(\frac{n\pi x}{b}\right)=$$

$$=\frac{c_1}{\sinh\left(\frac{n\pi x}{b}\right)}\left(\sinh\left(\frac{n\pi a}{b}\right)\cosh\left(\frac{n\pi x}{b}\right)-\cosh\left(\frac{n\pi a}{b}\right)\sinh\left(\frac{n\pi x}{b}\right)\right)=$$

$$=c_3\sinh\left(\frac{n\pi(a-x)}{b}\right)$$

+2/2

$\boxed{n=0}$

$$\left.\begin{array}{l} F(x)=c_4x+c_5 \\ F(a)=0=c_4a+c_5 \end{array}\right| \Rightarrow F(x)=c_4(x-a) \quad \text{(extra page)}$$

**Part b.** [8 points] Solve the Laplace equation on a rectangle

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < a,\ 0 < y < b,$$

with the following boundary conditions:

More than one non-homog. BC.

↳ Need to split the problem in two

$$\boxed{u(0,y) = 1,} \quad u(a,y) = 0, \quad 0 < y < b,$$

and

$$u_y(x,0), \quad \boxed{u_y(x,b) = 1,} \quad 0 < x < a.$$

Hint: Use Part a.

We split the problem in two: (Part a) and $\left\{ \begin{array}{l} v_{xx} + v_{yy} = 0 \\ v(0,y) = 0 = v(a,y) \\ v_y(x,0) = 0, v_y(x,b) = 1 \end{array} \right.$ +2

Following same steps as in a), we find that

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = -\lambda \quad \left\{ \begin{array}{l} F''(x) + \lambda F(x) = 0 \\ F(0) = F(a) = 0 \end{array} \right. \to \lambda = \left(\frac{n\pi}{a}\right)^2, n = 1, 2, \ldots \quad F(x) = \sin\left(\frac{n\pi x}{a}\right)$$ +2

BC: $F(0) = F(a) = 0$

$G'(0) = 0$

- $G''(y) - \left(\frac{n\pi}{a}\right)^2 G(y) = 0 \Rightarrow G(y) = c_1 \cosh\left(\frac{n\pi y}{a}\right) + c_2 \sinh\left(\frac{n\pi y}{a}\right)$

$G'(0) = 0 = c_2 \frac{n\pi}{a} \Rightarrow c_2 = 0$. +1

Thus,

$$w(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \cosh\left(\frac{n\pi y}{a}\right)$$ +1

$$v_y(x,b) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \frac{n\pi}{a} \sinh\left(\frac{n\pi b}{a}\right) = 1$$ +1 $\Rightarrow$

$$\Rightarrow B_n \frac{n\pi}{a} \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a} \int_0^a \sin\left(\frac{n\pi x}{a}\right) dx$$ +1 $\Rightarrow$ (* extra page)

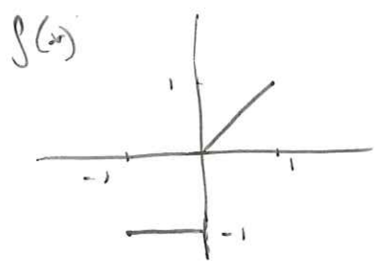
$$\Rightarrow B_n = \frac{2}{n\pi \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a \sin\left(\frac{n\pi x}{a}\right) dx. \quad (n = 1, 2, \ldots)$$

**Problem 4** [20 points]

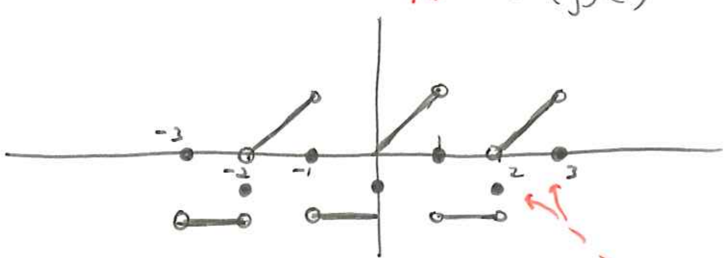
Consider the function $f : [-1,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} -1, & -1 \le x \le 0, \\ x, & 0 < x < 1. \end{cases}$$

**Part a.** [10 points] Sketch its Fourier series. Mark clearly the values at the points of discontinuity. What is the value of the series at $x = 0$? And at $x = 1$?




+5 $\mathcal{F}(f)(x)$




$x = 0 \to \mathcal{F}(f)(0) = \frac{-1}{2}$ +5 (together with)

$x = 1 \to \mathcal{F}(f)(1) = 0$ .

**Part b.** [10 points] Write the Fourier series of $f(x)$, and compute the coefficients. Simplify the results.

$$F(f)(x) = A_0 + \sum_{n=1}^{\infty}\left(A_n \cos(n\pi x) + B_n \sin(n\pi x)\right) \quad +3$$

$$A_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \frac{1}{2}\left(\int_{-1}^{0}(-1)\,dx + \int_0^1 x\,dx\right) = \frac{-1}{4} \quad +1$$

$$A_n = \int_{-1}^{1} f(x)\cos(n\pi x)\,dx \ (+1) = \int_{-1}^{0} -\cos(n\pi x)\,dx \ (+1) + \int_0^1 x\cos(n\pi x)\,dx =$$

$$= -\cancel{\frac{\sin(n\pi x)}{n\pi}}\Big]_{-1}^{0} + \frac{(-1)^n - 1}{n^2\pi^2} \ (+1) = \begin{cases} 0 & n \text{ even} \\ \frac{-2}{n^2\pi^2} & n \text{ odd} \end{cases} \quad +3$$

$$B_n = \int_{-1}^{1} f(x)\sin(n\pi x)\,dx \ (+1) = \int_{-1}^{0} -\sin(n\pi x)\,dx \ (+1) + \int_0^1 x\sin(n\pi x)\,dx = \quad +3$$

$$= \frac{1-(-1)^n}{n\pi} - \frac{\cancel{n\pi}(-1)^n}{n^{\cancel{2}}\pi^{\cancel{2}}} \ (+1) = \frac{1-(-1)^n-(-1)^n}{n\pi} = \begin{cases} \frac{-1}{n\pi}, & \text{even} \\ \frac{3}{n\pi}, & n \text{ odd} \end{cases}$$

0.5 if left with $\cos(n\pi)$

**Problem 5** [20 points]

**Part a.** [8 points] Let $f(x) = 3x$, and assume that the following expansion holds for $0 < x < \pi$,

$$f(x) = \sum_{n=0}^{\infty} \frac{b_n}{2^n} \sin(nx).$$

What is the value of $b_5$?

$$\frac{b_5}{2^5} = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(5x)\, dx = \frac{2}{\pi} \int_0^{\pi} 3x \sin(5x)\, dx \Rightarrow$$

$$\Rightarrow b_5 = \frac{2^6}{\pi} \frac{3\pi}{5} = \frac{3}{5} \cdot 2^6 .$$

+2 for demonstrating setup w/ orthogonality

+ (2 to 3) for sufficient effort

Full point for right answer + logical workflow

**Part b.** [6 points: 2 points each] Decide whether the following statements are true or false:

- The following PDE can be solved via the method of separation of variables:

$$u_t = u_{xxxx} + u_x + 5u.$$

T

- The even extension of a continuous function $f : [0,1] \to \mathbb{R}$ never has a jump discontinuity. T

- Consider the heat equation $u_t = u_{xx} + x$ on a one-dimensional rod of length $L$ subject to the boundary conditions

F

$$u(0,t) = 0, \quad u(L,t) = 0.$$

The sum of any two solutions (to the given heat equation and boundary conditions) is again a solution.

**Part c.** [6 points] If the function $v(x,y)$ solves $v_x - v = 0$ with $v(0,y) = y^2$, and $u(x,y)$ satisfies $u_x - u = 4y$, with $u(0,y) = 0$, find the solution $w(x,y)$ to $w_x - w = 12y$ with $w(0,y) = 2y^2$. Give the solution in terms of the functions $u$ and $v$.

$w = 3u + 2v$

Extra space for work:

(a) <u>Superposition:</u>
$$\begin{pmatrix} n=0: \ u(x,y) = C_0(x-a) \\ n\neq 0: \ u(x,y) = C_n \sinh\left(\frac{n\pi(a-x)}{b}\right)\cos\left(\frac{n\pi y}{b}\right) \end{pmatrix}$$

$\hookrightarrow$

+2/2
$$u(x,y) = C_0(x-a) + \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi(a-x)}{b}\right)\cos\left(\frac{n\pi y}{b}\right).$$

• <u>$u(0,y)=1$</u> :

+1
$$-aC_0 + \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi a}{b}\right)\cos\left(\frac{n\pi y}{b}\right) = 1 \Rightarrow$$

$$\Rightarrow \left\| \begin{array}{l} C_0 = \frac{-1}{a} \\ C_n = 0 \quad \forall n \neq 0 \end{array} \right.$$
+1

Therefore, $\left\| u(x,y) = \frac{-1}{a}(x-a) = 1 - \frac{x}{a} \right\|$.

(b) Thus, the solution to part b) is

$$\left\| \begin{array}{l} u(x,y) = 1 - \frac{x}{a} + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right)\cosh\left(\frac{n\pi y}{a}\right), \text{ with} \\ B_n = \frac{2}{n\pi \sinh\left(\frac{n\pi b}{a}\right)} \frac{a}{n\pi} \cos\left(\frac{n\pi x}{a}\right)\Big]_0^a = \frac{2a}{(n\pi)^2 \sinh\left(\frac{n\pi b}{a}\right)}\left((-1)^n - 1\right) \end{array} \right\|$$

Extra space for work:

Extra space for work:

Extra space for work: