HOMEWORK ASSIGNMENT 6, Math 425

Name: **Due:** Friday March 6, 8PM

For problems 1 to 4 you are not required to compute the coefficients of the series expansion.

Problem 1: Strauss, Section 4.2 #2, p.92

Consider the wave equation $u_{tt} = c^2 u_{xx}$ for $0 < x < L$, with boundary conditions $u_x(0,t) = u(L,t) = 0$ (Neumann at the left, Dirichlet at the right).

(a) Show that the eigenfunctions are $\cos\left((n+\frac{1}{2})\pi x/L\right)$.
(b) Write the series expansion for a solution $u(x,t)$.

Problem 2: Strauss, Section 4.2 #1, p.92

Solve the diffusion problem $u_t = ku_{xx}$ in $0 < x < L$, with the mixed boundary conditions $u(0,t) = u_x(L,t) = 0$.

Problem 3: Strauss, Section 4.1 #3, p.89

This problem shows another PDE that can be solved using separation of variables (see also Problem 3, Section 4.2, p.92, Strauss' book).

A quantum-mechanical particle on the line with an infinite potential outside the interval $(0,L)$ ("particle in a box") is given by Schrödinger's equation $u_t = iu_{xx}$ on $(0,L)$ with Dirichlet conditions at the ends (here i is the complex unit). Separate the variables to find the representation of its solution as a series.

Problem 4: Strauss, Section 4.2 #4, p.92

Consider the diffusion equation inside an enclosed circular tube. Let its length (circumference) be $2L$. Let x denote the arc length parameter where $-L < x < L$. Then the concentration of the diffusing substance satisfies

$$u_t = ku_{xx}, \qquad -L < x < L,$$
$$u(-L,t) = u(L,t), \qquad u_x(-L,t) = u_x(L,t).$$

These are called **periodic boundary conditions**.

(a) Show that the eigenvalues are $\lambda = (n\pi/L)^2$ for $n = 0, 1, 2, 3, \ldots$.

(b) Show that the concentration is

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \Big(A_n \cos\big(\frac{n\pi x}{L}\big) + B_n \sin\big(\frac{n\pi x}{L}\big) \Big) e^{-n^2\pi^2 kt/L^2}.$$

Problem 5:

Solve the following damped wave equation

$$u_{tt} = u_{xx} - 5\pi u_t, \quad 0 < x < 1, \quad t > 0,$$

with initial conditions

$$u(x,0) = 4 + 3\cos(\pi x) + \cos(2\pi x), \quad u_t(x,0) = 0,$$

and boundary conditions

$$u_x(0,t) = 0 = u_x(1,t).$$

Remark: Pay attention to the first three eigenvalues.

Problem 6: Strauss, Section 5.1 #9, p.112

Solve $u_{tt} = c^2 u_{xx}$ for $0 < x < \pi$, with the boundary conditions $u_x(0,t) = u_x(\pi,t) = 0$ and the initial conditions $u(x,0) = 0$, $u_t(x,0) = \cos^2 x$.

Problem 7: Strauss, Section 5.1 #4, p.111

Find the Fourier cosine series of the function $|\sin x|$ in the interval $(-\pi, \pi)$. Use it to find the sums

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}.$$

Problem 8: Strauss, Section 5.1 #5, p.112

Compute the Fourier sine series of $\phi(x) = x$ on $(0, L)$. Assume that the series can be integrated term by term, a fact that will be shown later.

(a) Find the Fourier cosine series of the function $x^2/2$. Find the constant of integration that will be the first term in the cosine series.
(b) Then by setting $x = 0$ in your results, find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

Problem 10 (optional): Strauss, Section 4.3 #1, p.100

For $a \neq 0$, find the eigenvalues graphically for the following boundary value problem

$$
\begin{aligned}
&-X'' = \lambda X, \\
&X(0) = 0, \quad X'(L) + aX(L) = 0.
\end{aligned}
$$

Problem 11 (optional): Strauss, Section 4.3 #9, p.100

On the interval $0 \leq x \leq 1$ of length one, consider the eigenvalue problem

$$
\begin{gathered}
-X'' = \lambda X \\
X'(0) + X(0) = 0 \qquad X(1) = 0,
\end{gathered}
$$

(absorption at one end and zero at the other).

(a) Find an eigenfunction with eigenvalue zero. Call it $X_0(x)$.
(b) Find an equation for the positive eigenvalues $\lambda = \beta^2$.
(c) Show graphically from part (b) that there are an infinite number of positive eigenvalues.
(d) Is there a negative eigenvalue?

Problem 12:

Read Sections 5.1, 5.2, 5.3, 5.4, and 5.5 of W. Strauss book.