HOMEWORK ASSIGNMENT 6, Math 241, Section 002

Name: **Due:** Friday March 6, 9pm

1. **[See Problem 4.4.2]** Consider the displacement $u(x,t)$ of a nonuniform string

$$\rho(x)u_{tt}(x,t) = T(x)u_{xx}(x,t) + Q(x,t).$$

Consider the case when $Q(x,t) = \alpha(x)u(x,t)$. Notice that, if $\alpha > 0$, this force tends to push the string further away from its unperturbed position $u = 0$, while if $\alpha < 0$ the body force is restoring.

   a) Separate variables if $T(x) = T$ is constant. Solve the time-dependent ordinary differential equation.

   b) Let $\rho(x) = \rho$, $\alpha(x) = \alpha$ be constants too. Solve the initial value problem for $\alpha < 0$ and

$$u(0,t) = 0 = u(L,t), \quad t > 0,$$
$$u(x,0) = 0, \quad u_t(x,0) = f(x), \quad 0 < x < L.$$

2. **[A variation of Problem 4.4.9]** Consider the damped wave equation

$$u_{tt} = u_{xx} - u_t, \quad 0 < x < L, \quad t > 0,$$

and boundary conditions $u_x(0,t) = 0$, $u_x(L,t) = 0$.

   a) Show that the total energy

$$E(t) = \int_0^L \frac{1}{2}(u_t(x,t))^2 dx + \int_0^L \frac{1}{2}(u_x(x,t))^2 dx$$

   is a non-increasing function of time.

   b) If $u(x,0) = 0, u_t(x,0) = 0$, what can you conclude about $u(x,t)$?

   c) Show that the problem with initial conditions $u(x,0) = f(x)$, $u_t(x,0) = g(x)$ has a unique solution.
   Hint: Proceed by contradiction as follows: Assume u and v are two different solutions. Study the energy for $w = u - v$.

3. **[A variation of 4.4.5]** Solve the following damped wave equation

$$u_{tt} = u_{xx} - 5\pi u_t, \quad 0 < x < 1, \quad t > 0,$$

with initial conditions

$$u(x,0) = 4 + 3\cos(\pi x) + \cos(2\pi x), \quad u_t(x,0) = 0,$$

and boundary conditions

$$u_x(0,t) = 0 = u_x(1,t).$$

Remark: Pay extra attention to the first three eigenvalues.

4. Compute the coefficients of the Fourier series over $[-\pi, \pi]$ of $f(x) = |\sin x|$ in its *complex form*. (Check Lecture 8 notes, page 5)

5. Read Sections 5.1, 5.2, 5.3 and 5.4 of R. Haberman's book.