HOMEWORK ASSIGNMENT 5, Math 425

Name: **Due:** Sunday March 1, 8PM

Remark: This homework is optional. It will count up to a maximum of 10 extra points towards the final cumulative grade.

Problem 1:

Find the solution to the following initial value problem:

$$\begin{aligned} u_{xx} - 3u_{xt} + 2u_{tt} &= (4x+2t)(u_x - 2u_t) + t + 2x, \quad x \in \mathbb{R}, t > 0, \\ u(x,0) &= 0, \\ u_t(x,0) &= 0. \end{aligned}$$

Write the solution in terms of the function $\mathcal{E}(x) = \displaystyle\int_0^x e^{-s^2} ds$.

Problem 2:

Consider the modified *damped* wave equation on the real line:

$$\begin{aligned} &u_{tt} = u_{xx} - \alpha u_t - \beta u, \quad -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = \phi(x), \\ &u_t(x,0) = \psi(x). \end{aligned}$$

a) For $\beta = 0$, $\alpha > 0$, show that the following energy is non-increasing for all time

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} \Big(|u_t(x,t)|^2 + |u_x(x,t)|^2 \Big) dx.$$

b) For $\alpha = 0$, $\beta > 0$, can we guarantee that $E(t)$ is non-increasing for all time? Explain. Hint: Maybe do Part c) first.

c) For $\alpha = 0$, $\beta > 0$, find a non-negative quantity that is preserved in time.

d) Show that for $\alpha > 0$, $\beta > 0$, the solution has a unique solution.

Problem 3:

Consider the one-dimensional heat equation in a rod with a source

$$u_t = u_{xx} + x^3, \quad 0 < x < L, \quad t > 0, \tag{1}$$

with the left end of the rod, $x = 0$, insulated, and $u_x(L, t) = \beta$. The initial temperature distribution is $u(x, 0) = f(x)$.

a) Find the total physical energy $E(t) = \int_0^L u(x, t)dx$ as a function of time.

b) Find the value of β that makes $E(t)$ be constant in time.

c) Show that for $\beta = -L^4/4$ there exist time-independent solutions. That is, functions not depending on time $v(x)$ that solve the (1).

d) Find the time-independent solution that has the same physical energy as the initial data. This is called the equilibrium solution of the problem (and it is indeed the temperature distribution towards which the time-dependent solution converges as time goes to infinity).

Problem 4:

Consider the following inhomogeneous heat equation with dissipation

$$\begin{aligned} u_t &= 4u_{xx} - 3u + e^{-x}, \quad -\infty < x < \infty, \quad t > 0, \\ u(x, 0) &= 0. \end{aligned}$$

a) Show that there is a unique solution to the initial value problem.

b) By *direct substitution* into the equation (with care), show that the solution is given by

$$u(x, t) = \int_0^t \int_{-\infty}^{\infty} S(x - y, t - \tau)e^{-y-3(t-\tau)}dyd\tau,$$

where $S(x, t)$ is the heat kernel.