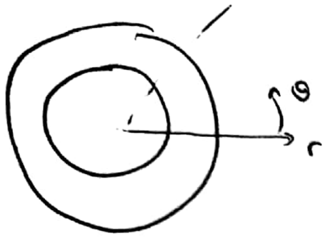
$\boxed{\text{P16}}$



$$\left.\begin{array}{l} \Delta u = 0 \quad , \ 1<r<2, \ -\pi \le \theta \le \pi \\ u_r(1,\theta)=0 \\ u_r(2,\theta) = 1 + c\sin^2(3\theta) \end{array}\right\}$$

a) From a physical viewpoint, solutions to the Laplace equation correspond to equilibrium solutions of the heat equation $u_t = \Delta u$. Therefore, the net flux must be zero so that the total energy is constant in time.

Therefore,

$$0 = \int_0^{2\pi} (1 + c\sin^2(3\theta))\, R_2\, d\theta =$$

$$= R_2\left(2\pi + c\int_0^{2\pi} \frac{1-\cos(6\theta)}{2}\, d\theta\right) = 2(2\pi + c\pi) = 0 \Leftrightarrow$$

$$\Leftrightarrow \boxed{c = -2}$$

• The same conclusion can be obtained mathematically:

$$\Delta u = 0 \Rightarrow \iint_A \Delta u \, dS = 0 \Rightarrow \oint_{\partial A} \vec{n}\cdot\nabla u \, dl = 0 \quad \left(\begin{array}{l}\text{integration by} \\ \text{parts}\end{array}\right).$$

The line integral is:

$$\oint_{\partial A} \vec{n}\cdot\nabla u\, dl = \int_0^{2\pi} (-u_r(\cancel{1},\theta)) R_1\, d\theta + \int_0^{2\pi} u_r(2,\theta)\, R_2\, d\theta = 0 \ //.$$

b) We apply separation of variables:

$u(r,\theta) = F(r)G(\theta) \rightsquigarrow$

$$\rightsquigarrow \frac{r}{F(r)}\frac{d}{dr}\left(rF'(r)\right) = -\frac{G''(\theta)}{G(\theta)} = \lambda \rightarrow$$

$\rightarrow$ ① $G''(\theta) + \lambda G(\theta) = 0$ ② $\frac{r}{F(r)}\frac{d}{dr}(rF'(r)) = \lambda$.

<u>BC:</u>

Periodic: $G(-\pi) = G(\pi)$
$G'(-\pi) = G'(\pi)$ (from $u(r,-\pi) = u(r,\pi)$, $u_\theta(r,-\pi) = u(r,\pi)$).

Given: $F'(1) = 0$ (from $u_r(1,\theta) = 0$).

Thus,

$$\left.\begin{array}{l} ① \; G''(\theta) + \lambda G(\theta) = 0 \\ G(-\pi) = G(\pi) \\ G'(-\pi) = G'(\pi) \end{array}\right\} \xrightarrow{\text{(table)}} \begin{array}{l} \lambda = n^2, \; n = 0, 1, \ldots \\ G(\theta) = c_1\cos(n\theta) + c_2\sin(n\theta). \end{array}$$

② $\frac{r}{F(r)}\frac{d}{dr}(rF'(r)) = n^2 \rightarrow$ <u>$n=0$</u>: $F(r) = c_3\ln(r) + c_4$
$F'(1) = c_3 = 0$

<u>$n>0$</u>: $F(r) = r^p$
$p = \pm n \rightarrow F(r) = c_5 r^n + c_6 r^{-n}$
$F'(1) = c_5 n - c_6 n = 0 \rightarrow c_5 = c_6$

②

$$F(r) = c\left(r^n + r^{-n}\right), \quad n = 0, 1, \dots$$

- Superposition principle:

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n\left(r^n + r^{-n}\right)\cos(n\theta) + \sum_{n=1}^{\infty} B_n\left(r^n + r^{-n}\right)\sin(n\theta),$$

- Nonhomogeneous BC:

$$u_r(2,\theta) = 1 - 2\sin^2(3\theta) = 1 - \left(1 - \cos(6\theta)\right) = \cos(6\theta)$$

$$\hookrightarrow \cos(6\theta) = \sum_{n=1}^{\infty}\left(A_n n\left(2^{n-1} - 2^{-n-1}\right)\cos(n\theta) + B_n n\left(2^{n-1} - 2^{-n-1}\right)\sin(n\theta)\right)$$

$$\hookrightarrow A_6 \cdot 6\left(2^5 - 2^{-7}\right) = 1 \Rightarrow A_6 = \frac{1}{6\left(32 - \frac{1}{128}\right)} = \frac{2^7}{6\left(2^{12} - 1\right)}$$

$A_n = 0$ for $n \neq 6, 0$.

$B_n = 0$ for all n.

<u>Remark</u>: The solution is

$$u(r,\theta) = 2A_0 + \frac{1}{6}\,\frac{r^6 + r^{-6}}{2^5 + 2^{-7}}\cos(6\theta), \text{ with } A_0 \in \mathbb{R} \text{ any constant.}$$

(solution non uniquely defined).