Name: ______________________________ **PennID**: ______________

Math 425/AMCS 525
Math 425 - Midterm 1 - Practice Exam
February 20, 2020, 1:30-2:50pm

Please *turn off and put away all electronic devices*. You are allowed to use one side of a 8x11 cheat-sheet with hand-written notes during this exam. No calculators, no books. Read the problems carefully. **Show all work** (answers without proper justification will not receive full credit). Be as organized as possible: illegible work will not be graded. Please sign and date the pledge below to comply with the Code of Academic Integrity. Don't forget to write your Name and PennID on the top of this page. Good luck!

#	Points possible	Your score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination.

______________________________ ______________

Signature Date

First-order Linear PDEs: $a(x,y)u_x(x,y) + b(x,y)u_y(x,y) + c(x,y)u(x,y) = f(x,y)$

- Method of Characteristics: Noticing that $au_x + bu_y = (a,b) \cdot \nabla u$, the PDE is transformed into an ODE along an appropriate curve.
  Remark: This curve cannot always be parametrized as $y = y(x)$.

- Method of Coordinates: Look for an adequate change of variables. When a and b are constants, a good change is $\xi = ax + by$, $\eta = bx - ay$.

Second-order Linear PDEs: Heat equation, Wave equation, Laplace equation (and their variations).

- Wave equation:

  1. $u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty \to u(x,t) = f(x+ct) + g(x-ct).$
  2. $u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x) \to$ D'Alembert

$$u(x,t) = \frac{1}{2}\big(\phi(x+ct) - \phi(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds.$$

- Heat equation:

  1. $u_t = ku_{xx}, \quad u(x,0) = H(x), \quad -\infty < x < \infty, \quad (H(x)$ Heaviside function)

$$u(x,y) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-s^2} ds.$$

  2. $u_t = ku_{xx}, \quad u(x,0) = \phi(x), \quad -\infty < x < \infty,$

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy, \quad S(x,t) = \frac{1}{\sqrt{4k\pi t}} e^{-\frac{-x^2}{4kt}}.$$

  3. $u_t = ku_{xx} + f(x,t), \quad u(x,0) = \phi(x), \quad -\infty < x < \infty,$

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-\tau)f(y,\tau)dyd\tau.$$

  4. $u_t = ku_{xx}, \quad u(x,0) = \phi(x),\ u(0,t) = 0, \quad 0 < x < \infty,$

$$u(x,t) = \int_0^{\infty} (S(x-y,t) - S(x+y))\phi(y)dy.$$

  5. $u_t = ku_{xx}, \quad u(x,0) = \phi(x),\ u_x(0,t) = 0, \quad 0 < x < \infty,$

$$u(x,t) = \int_0^{\infty} (S(x-y,t) + S(x+y))\phi(y)dy.$$

Problem 1 (20 pts)

Solve

$$3u_{xx} + 10u_{xt} + 3u_{tt} = 0,$$
$$u(x,0) = e^{-x}\sin x, \quad u_t(x,0) = \frac{1}{1+x^2}.$$

Solution (Problem 1):

Solution (Problem 1):

Problem 2 (20 pts)

Use the method of characteristics to solve

$$\sin(y)u_x + 2u_y + u = 1,$$
$$u(0,y) = \cos(y),$$

and find the region of the xy plane in which the solution is uniquely determined.

Solution (Problem 2):

Solution (Problem 2):

Problem 3 (20 pts): Give a proof of the weak maximum principle for the Laplace equation on a rectangle:

$$u_{xx} + u_{yy} = 0, \quad 0 \leq x \leq a, \quad 0 \leq y \leq b.$$

The weak maximum principle here says: if $u(x, y)$ is a solution, then the maximum of $u(x, y)$ on the whole rectangle $0 \leq x \leq a$, $0 \leq y \leq b$ is equal to the maximum of $u(x, y)$ on the boundaries (i.e., on $x = 0$ or $x = a$ or $y = 0$ or $y = b$).

Hint: Proceed as in the heat equation, by defining an auxiliary function $v(x, y) = u(x, y) + \varepsilon|x|^2 = u(x, y) + \varepsilon(x^2 + y^2)$ $(\varepsilon > 0)$.

Solution (Problem 3):

Solution (Problem 3):

Problem 4 (20 pts) Consider the following heat equation on the real line

$$u_t = 3u_{xx} - u, \quad -\infty < x < \infty,\ t > 0,$$
$$u(x,0) = e^{-|x|}.$$

Part a. Prove that the initial value problem has a unique solution.

Part b. Show that the solution to the problem is an even function in x for all time $t \geq 0$. You do not need to find the solution.

Solution (Problem 4):

Problem 5 (20 pts) Solve the following inhomogeneous diffusion problem on the half-line:

$$\begin{aligned} &u_t - ku_{xx} = f(x,t), \qquad \text{for } 0 < x < \infty, t > 0, \\ &u_x(0,t) = h(t), \qquad \text{for } 0 < t \\ &u(x,0) = \phi(x), \qquad \text{for } 0 < x < \infty. \end{aligned}$$

You can use when needed any formula from the cheat-sheet provided on page 2. All other steps must be shown.

Hint: Define a new function $v(x,t)$ in such a way that it satisfies a new problem with an homogeneous boundary condition $v_x(0,t) = 0$.

Solution (Problem 5):

Solution (Problem 5):

Extra paper

Extra paper

Extra paper