Name ______________________________

Math 241 - Section 002 - Midterm 1 - Practice exam
Thursday, February 20, 2020, @ 10:30 AM - 11:50 AM

No form of cheating will be tolerated. You are expected to uphold the Code of Academic Integrity. I certify that all of the work on this test is my own.

Signature: Eduardo García-Juárez

The use of computers, smartphones, and similar devices is neither necessary nor permitted during this exam. Correct answers without proper justification will not receive full credit. Clearly highlight your answers and the steps taken to arrive at them: illegible work will not be graded. No cheat-sheet is allowed.

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Problem	Points	Your score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Problem 1 [25 points]

This problems deals with the diffusion of a chemical pollutant on a straight pipe of length L, with constant sectional area. Let $u(x,t)$ be the density or concentration of the chemical per unit volume.

Part a. [4 points] State in mathematical terms the conservation of mass of the pollutant in an arbitrary region between $x = a$ and $x = b$.

(this is enough)

For any times t and $t+h > t$,

$$A\int_a^b u(x,t+h)\,dx = A\int_a^b u(x,t)\,dx + A\int_t^{t+h}\phi(a,\tau)\,d\tau - A\int_t^{t+h}\phi(b,\tau)\,d\tau,$$ where

ϕ is the mass flux. Taking the limit as $h \to 0^+$, we obtain that

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = \phi(a,t) - \phi(b,t).$$

Part b. [4 points] Assuming that Fick's law of diffusion is satisfied,

$$\phi = -k\frac{\partial u}{\partial x},$$

show that the partial differential equation that models the evolution of $u(x,t)$ is

$$u_t = ku_{xx}.$$

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = -\int_a^b \phi_x(x,t)\,dx \Rightarrow$$

$$\Rightarrow u_t(x,t) = -\phi_x(x,t) = k\,u_{xx}(x,t) \,//.$$

Part c. [6 points] Assume that the initial condition is $u(x,0) = f(x)$ and that the mass flux is specified at both ends $-ku_x(0,t) = \alpha$, $-ku_x(L,t) = 3$. Determine the total amount of chemical in the pipe as a function of time (write the solution in terms of the constant α):

$$M(t) = A\int_0^L u(x,t)dx.$$

$$M'(t) = A\int_0^L u_t(x,t)dx = A\int_0^L k u_{xx}(x,t)\,dx = A\,k u_x\Big]_0^L =$$

$$= A\left(ku_x(L,t) - ku_x(0,t)\right) = A(-3+\alpha) \Rightarrow$$

$$\Rightarrow M(t) = M(0) + A(\alpha-3)t = A\int_0^L f(x)dx + A(\alpha-3)t.$$

Part d. [6 points] Under what conditions is there an equilibrium chemical concentration, and what is it?

We need $M'(t) = 0$ (otherwise there is a change in the mass, and thus of $u(x,t)$).

That is, $\boxed{\alpha = 3}$.

Equilibrium: $u(x,t) = u(x) \rightarrow u_t = 0$.

We have to solve $u_{xx}(x) = 0 \Rightarrow u(x) = c_1 x + c_2$.

B.C.: $\left.\begin{array}{l} -ku_x(0) = 3 \\ -ku_x(L) = 3 \end{array}\right\}$ $u_x(x) = c_1 \Rightarrow c_1 = -\dfrac{3}{k}$.

To find c_2, we use the conservation of mass:

$$A\int_0^L u(x)dx = A\int_0^L \left(\frac{-3}{k}x + c_2\right)dx = A\int_0^L f(x)dx \Rightarrow c_2 = \frac{3L}{2k} + \frac{1}{L}\int_0^L f(x)dx.$$

Problem 2 [20 points]

Solve the heat equation $u_t = 4u_{xx}$ for $-2 < x < 2$, subject to the following boundary and initial conditions:

$$u(-2,t) = u(2,t), \quad u_x(-2,t) = u_x(2,t) = 0,$$
$$u(x,0) = 5\sin(2\pi x) + \cos(\pi x).$$

$u(x,t) = F(x)G(t)$

$\hookrightarrow \quad G'(t)F(x) = 4F''(x)G(t) \rightarrow \frac{1}{4}\frac{G'(t)}{G(t)} = \frac{F''(x)}{F(x)} = -\lambda \rightarrow$

$\Rightarrow$ ① $\left.\begin{array}{l} F''(x) + \lambda F(x) \\ F(-2) = F(2) \\ F'(-2) = F'(2) \end{array}\right\} \xrightarrow[\text{B.C.}]{\text{periodic}}$ $\lambda = \left(\frac{n\pi}{2}\right)^2$

$F(x) = c_1 \cos\left(\frac{n\pi x}{2}\right) + c_2 \sin\left(\frac{n\pi x}{2}\right)$.

② $G'(t) = e^{-\left(\frac{n\pi}{2}\right)^2 4t}$

Therefore, $u(x,t) = \sum_{n=0}^{\infty} \left(A_n \cos\left(\frac{n\pi x}{2}\right) + B_n \sin\left(\frac{n\pi x}{2}\right)\right) e^{-4\left(\frac{n\pi}{2}\right)^2 t}$.

Using the initial data,

$B_4 = 5$, $A_2 = 1$, all the other coefficients are zero.

Problem 3 [20 points]

Consider the 2D heat equation in a quarter of an annulus with inner radius $R_1 = 1$ and outer radius $R_2 = 2$, $\mathcal{A} = \{(r,\theta) : 0 \leq \theta \leq \pi/2, 1 \leq r \leq 2\}$. Together with the boundary and initial conditions, the problem is the following:

$$
\begin{aligned}
u_t = \Delta u, \quad 0 < \theta < \frac{\pi}{2},\ 1 < r < 2 \\
u_r(1,\theta,t) = C, \quad u_r(2,\theta,t) = \cos(2\theta), \\
u_\theta(r,\pi/2,t) = 0, \qquad u_\theta(r,0,t) = 0,
\end{aligned}
$$

where

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

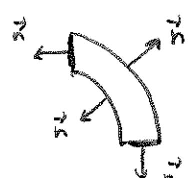
Part a. [4 points] Define the total thermal energy by $E(t) = \iint_{\mathcal{A}} u\, dS$. Show that

$$E'(t) = -\frac{\pi}{2}C.$$

$$\iint_A u_t\, dS = \iint_A \Delta u\, dS \Rightarrow \frac{d}{dt}\iint_A u\, dS = \int_{\partial A} \vec{n}\cdot\nabla u\, dl \Rightarrow$$

$$\Rightarrow E'(t) = \int_{\partial A} \vec{n}\cdot\nabla u\, dl.$$

Domain:



$$E'(t) = \int_{\partial A} \vec{n}\cdot\nabla u\, dl = \int_0^{\pi/2} 2\, u_r(2,\theta,t)\, d\theta + \int_1^2 u_\theta(\cancel{r}, \tfrac{\pi}{2}, t)\, dr - \int_0^{\pi/2} u_r(1,\theta,t)\, d\theta +$$

$$- \int_1^2 u_\theta(\cancel{r}, 0, t)\, dr = 2\int_0^{\pi/2} \cos(2\theta)\, d\theta - C\frac{\pi}{2} = -\frac{\pi}{2}C$$

Part b. [2 points] Briefly explain why the condition $C = 0$ is needed for an equilibrium solution to exist.

From part a), $C=0$ makes the total energy preserved in time.

Part c. [8 points] For $C = 0$, find an equilibrium solution to the problem. Explain why the solution is not uniquely defined.

$u(r,\theta) \rightsquigarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$. Let $u(r,\theta) = F(r)G(\theta)$.

Then, BC: $F'(1) = 0$, $G'(\frac{\pi}{2}) = 0$, $G'(0) = 0$.

① $$\left.\begin{array}{l} G''(\theta) + \lambda G(\theta) = 0 \\ G'(\frac{\pi}{2}) = G'(0) = 0 \end{array}\right| \rightsquigarrow \left\{\begin{array}{l} \lambda = \left(\frac{n\pi}{\pi/2}\right)^2 = (2n)^2, \ n = 0, 1, \ldots \\ G(\theta) = \cos(2n\theta) \end{array}\right.$$

② $$\left.\begin{array}{l} r^2 F''(r) + rF(r) - \left(\frac{n\pi}{\pi/2}\right)^2 F(r) = 0 \\ F'(1) = 0 \end{array}\right\}$$

$\underline{n=0}$: $$\left.\begin{array}{l} F(r) = c_1 \ln r + c_2 \\ F'(1) = c_1 = 0 \end{array}\right\} \rightsquigarrow F(r) = c_2.$$

$\underline{n>0}$: Try with $F(r) = r^p \rightsquigarrow p = \pm 2n$
(Euler equation)

$$F(r) = c_3 r^{2n} + c_4 r^{-2n}$$

$$F'(1) = 2n c_3 1^{2n-1} - 2n c_4 1^{-2n-1} = 2n c_3 - 2n c_4 = 0 \Rightarrow c_3 = c_4.$$

Thus, $u(r,\theta)=\sum_{n=0}^{\infty} A_n\cos(2n\theta)\left(r^{2n}+\frac{1}{r^{2n}}\right)$.

Finally, $u_r(2,\theta)=\sum_{n=1}^{\infty} A_n\cos(2n\theta)2n\left(2^{2n-1}-\frac{1}{2^{2n+1}}\right)=\cos(2\theta)\Rightarrow$

$$\left.\begin{array}{l}\Rightarrow A_1 2\left(2-\frac{1}{2^3}\right)=1\\ A_n=0 \text{ for } n\geq 2\end{array}\right\}\Rightarrow \begin{array}{l}A_1=\frac{4}{15}\\ A_n=0 \text{ for } n\geq 2.\end{array}$$

- Therefore, $u(r,\theta)=\underset{\substack{\uparrow\\ \text{undetermined.}}}{2A_0}+\frac{4}{15}\cos(2\theta)\left(r^2+\frac{1}{r^2}\right)$.

Part d. [6 points] Assuming that we are giving the additional information $E(0)$, determine the unique equilibrium solution.

$E'(t)=0\Rightarrow E(t)=E(0)$ for all $t\geq 0$. In particular, at equilibrium. Thus,

$$\iint_A u_{\text{equil.}}\,dS=E(0)$$

$$\int_0^{\pi/2}\int_1^2 u(r,\theta)\,r\,dr\,d\theta=\int_0^{\pi/2}\int_1^2 2A_0\,r\,dr\,d\theta+\frac{4}{15}\int_0^{\pi/2}\int_1^2\cos(2\theta)\left(r^2+\frac{1}{r^2}\right)r\,dr\,d\theta=$$

$$=2A_0\frac{\pi}{2}\left(\frac{4}{2}-\frac{1}{2}\right)+\frac{4}{15}\int_0^{\pi/2}\cancel{\cos(2\theta)\,d\theta}\int_1^2\left(r^2+\frac{1}{r^2}\right)r\,dr\Rightarrow$$

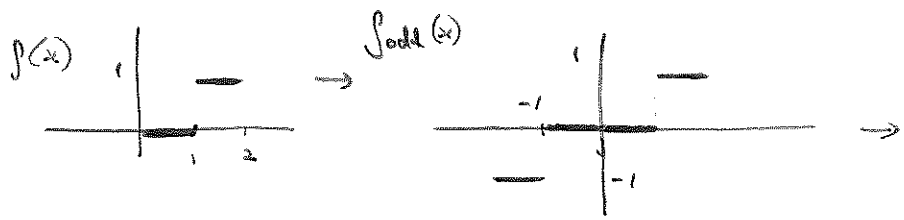
$$\Rightarrow \pi A_0\frac{3}{2}=E(0)\Rightarrow A_0=\frac{2}{3}\pi E(0)$$

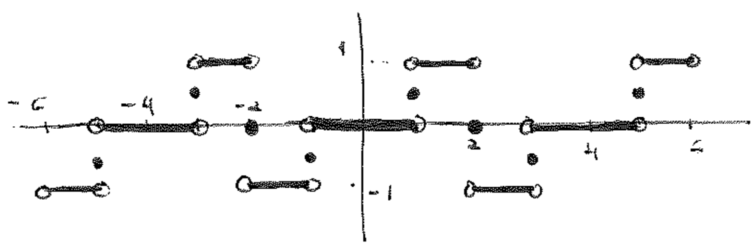
Problem 4 [20 points]

Consider the function $f : [0, 2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & x < 1, \\ 1, & x > 2 \end{cases}$$

Part a. [10 points] Sketch its Fourier sine series. Mark clearly the values at the points of discontinuity. What is the value of the series at $x = 0$? And at $x = -1$?





At $x = 0$ converges to 0.

At $x = -1$ converges to $\frac{-1}{2}$.

Part b. [5 points] Write the Fourier sine series of $f(x)$ over $[0, 2]$, and compute the coefficients. Simplify the results.

$$S(f)(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{2}\right), \text{ with}$$

$$B_n = \frac{2}{2}\int_0^2 f(x) \sin\left(\frac{n\pi x}{2}\right) = \int_1^2 \sin\left(\frac{n\pi x}{2}\right) dx = \left(-\frac{\cos\left(\frac{n\pi x}{2}\right)}{n\pi/2}\right]_1^2 =$$

$$= \frac{2}{n\pi}\left(-\cos(n\pi) + \cos\left(\frac{n\pi}{2}\right)\right) =$$

$$= \begin{cases} \frac{1}{k\pi}\left(-1 + (-1)^k\right) & \text{for } n = 2k \\ \frac{2}{n\pi} & \text{for } n \text{ odd.} \end{cases}$$

Problem 5 [20 points]

Part a. [10 points] Let $f(x) = -4x$, and assume that the following expansion holds for $0 < x < 2$,

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{2^n} \cos\left(\frac{n\pi x}{2}\right).$$

What are the values of a_0 and a_4?

$$\int_0^2 f(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n}{2^n}\int_0^2 \cos\left(\frac{n\pi x}{2}\right)dx = \frac{a_0}{2^0}\,2 \Rightarrow a_0 = \frac{1}{2}\int_0^2 f(x)\,dx =$$

$$= \frac{1}{2}(-2\cdot 4) = -4.$$

$$\int_0^2 f(x)\cos\left(\frac{4\pi x}{2}\right)dx = \sum_{n=0}^{\infty}\frac{a_n}{2^n}\int_0^2 \cos\left(\frac{n\pi x}{2}\right)\cos\left(\frac{4\pi x}{2}\right)dx =$$

$$= \frac{a_4}{2^4}\int_0^2 \cos^2\left(\frac{4\pi x}{2}\right)dx = \frac{a_4}{16} \Rightarrow$$

$$\Rightarrow a_4 = -16\int_0^2 4x\cos\left(\frac{4\pi x}{2}\right)dx = \left\{\begin{array}{ll} u = x & \rightarrow du = dx \\ dv = \cos(2\pi x)\,dx & \rightarrow v = \frac{1}{2\pi}\sin(2\pi x) \end{array}\right\}$$

$$= -64\left(\cancel{\frac{x}{2\pi}\sin(2\pi x)\Big]_0^2} - \int_0^2 \frac{1}{2\pi}\sin(2\pi x)\,dx\right) =$$

$$= 64\,\frac{1}{2\pi}\,\frac{-\cos(2\pi x)}{2\pi}\Big]_0^2 = \frac{16}{\pi^2}(-1+1) = 0.$$

Part b. [5 points] Suppose that $u(x,t)$ solves

$$u_t = u_{xx} + u + \cos t, \quad 0 < x < L, \ \ t > 0,$$

with

$$u(0,t) = 0, \quad u_x(0,t) = 0,$$

and $u(x,0) = 0$. Suppose also that $v(x,t)$ solves

$$v_t = v_{xx} + v,$$

with

$$v(0,t) = f(t), \quad v_x(0,t) = 0,$$

and $v(x,0) = \cos x$. Find the solution $w(x,t)$ to the following initial boundary value problem

$$w_t = w_{xx} + w + 2\cos t, \quad 0 < x < L, \ \ t > 0,$$

with

$$w(0,t) = 4f(t), \quad w_x(0,t) = 0,$$

and $w(x,0) = 4\cos x$. Write the solution in terms of u and v.

$w(x,t) = 2u(x,t) + 4v(x,t)$.

Easy to check: follows by linearity.

Part c. [5 points] Consider the following BVP posed for $0 < x < L$ and $t > 0$:

$$u_t = u_{xx} + 2u_x + u,$$
$$u(0,t) = 0, \quad u(L,t) + u_x(L,t) = 0.$$

Apply the method of separation of variables to determine what ordinary differential equations are implied for the functions of x and t and what boundary conditions are necessary for each of those ODEs. You do not need to solve these ODEs.

$$u(x,t) = F(x)G(t)$$

$$F(x)\,G'(t) = F''(x)\,G(t) + 2F'(x)G(t) + F(x)\,G(t) \Rightarrow$$

$$\Rightarrow \frac{G'(t)}{G(t)} = \frac{F''(x)}{F(x)} + 2\frac{F'(x)}{F(x)} + 1 = \lambda \Rightarrow$$

$$\Rightarrow (1)\ G'(t) = \lambda G(t) \qquad (2)\ F''(x) + 2F'(x) + F(x) = \lambda F(x).$$

$\underline{BC:}$ $u(0,t) = 0 \Rightarrow F(0) = 0.$

$u(L,t) + u_x(L,t) = 0 \Rightarrow F(L) + F'(L) = 0.$