12 Heat conduction on the half-line

In previous lectures we completely solved the initial value problem for the heat equation on the whole line, i.e. in the absence of boundaries. Next, we turn to problems with physically relevant boundary conditions. Let us first add a boundary consisting of a single endpoint, and consider the heat equation on the half-line $D = (0, \infty)$. The following initial/boundary value problem, or IBVP, contains a Dirichlet boundary condition at the endpoint $x = 0$.

$$\begin{cases} v_t - kv_{xx} = 0, & 0 < x < \infty, 0 < t < \infty, \\ v(x,0) = \phi(x), & x > 0, \\ v(0,t) = 0, & t > 0. \end{cases} \tag{12.1}$$

If the solution to the above mixed initial/boundary value problem exists, then we know that it must be unique from an application of the maximum principle. In terms of the heat conduction, one can think of v in (12.1) as the temperature in an infinite rod, one end of which is kept at a constant zero temperature. The initial temperature of the rod is then given by $\phi(x)$.

Our goal is to solve the IBVP (12.1), and derive a solution formula, much like what we did for the heat IVP on the whole line. But instead of constructing the solution from scratch, it makes sense to try to reduce this problem to the IVP on the whole line, for which we already have a solution formula. This is achieved by extending the initial data $\phi(x)$ to the whole line. We have a choice of how exactly to extend the data to the negative half-line, and one should try to do this in such a fashion that the boundary condition of (12.1) is automatically satisfied by the solution to the IVP on the whole line that arises from the extended data. This is the case, if one chooses the *odd extension* of $\phi(x)$, which we describe next.

By the definition a function $\psi(x)$ is odd, if $\psi(-x) = -\psi(x)$. But then plugging in $x = 0$ into this definition, one gets $\psi(0) = 0$ for any odd function. Recall also that the solution $u(x,t)$ to the heat IVP with odd initial data is itself odd in the x variable. This follows from the fact that the sum $[u(x,t) + u(-x,t)]$ solves the heat equation and has zero initial data, hence, it is the identically zero function by the uniqueness of solutions. Then, by our above observation for odd functions, we would have that $u(0,t) = 0$ for any $t > 0$, which is exactly the boundary condition of (12.1).

This shows that if one extends $\phi(x)$ to an odd function on the whole line, then the solution with the extended initial data automatically satisfies the boundary condition of (12.1). Let us then define

$$\phi_{\text{odd}}(x) = \begin{cases} \phi(x) & \text{for } x > 0, \\ -\phi(-x) & \text{for } x < 0, \\ 0 & \text{for } x = 0. \end{cases} \tag{12.2}$$

It is clear that ϕ_{odd} is an odd function, since we defined it for negative x by reflecting the $\phi(x)$ with respect to the vertical axis, and then with respect to the horizontal axis. This procedure produces a function whose graph is symmetric with respect to the origin, and thus it is odd. One can also verify this directly from the definition of odd functions. Now, let $u(x,t)$ be the solution of the following IVP on the whole line

$$\begin{cases} u_t - ku_{xx} = 0, & -\infty < x < \infty, 0 < t < \infty, \\ u(x,0) = \phi_{\text{odd}}(x). \end{cases} \tag{12.3}$$

From previous lectures we know that the solution to (12.3) is given by the formula

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi_{\text{odd}}(y)\, dy, \qquad t > 0. \tag{12.4}$$

Restricting the x variable to only the positive half-line produces the function

$$v(x,t) = u(x,t)\big|_{x \geq 0}. \tag{12.5}$$

We claim that this $v(x,t)$ is the unique solution of IBVP (12.1). Indeed, $v(x,t)$ solves the heat equation on the positive half-line, since so does $u(x,t)$. Furthermore,

$$v(x,0) = u(x,0)\Big|_{x>0} = \phi_{\text{odd}}(x)\Big|_{x>0} = \phi(x),$$

and $v(0,t) = u(0,t) = 0$, since $u(x,t)$ is an odd function of x. So $v(x,t)$ satisfies the initial and boundary conditions of (12.1).

Returning to formula (12.4), we substitute the expressions for ϕ_{odd} from (12.2) and write

$$\begin{aligned} u(x,t) &= \int_0^\infty S(x-y,t)\phi_{\text{odd}}(y)\,dy + \int_{-\infty}^0 S(x-y,t)\phi_{\text{odd}}(y)\,dy \\ &= \int_0^\infty S(x-y,t)\phi(y)\,dy - \int_{-\infty}^0 S(x-y,t)\phi(-y)\,dy. \end{aligned}$$

Making the change of variables $y \mapsto -y$ in the second integral on the right, and flipping the integration limits gives

$$u(x,t) = \int_0^\infty S(x-y,t)\phi(y)\,dy - \int_0^\infty S(x+y,t)\phi(y)\,dy.$$

Using (12.5) and the above expression for $u(x,t)$, as well as the expression of the heat kernel $S(x,t)$, we can write the solution formula for the IBVP (12.1) as follows

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt}\right] \phi(y)\,dy. \tag{12.6}$$

The method used to arrive at this solution formula is called the *method of odd extensions* or the *reflection method*. We can make a physical sense of formula (12.6) by interpreting the integrand as the contribution from the point y minus the heat loss from this point due to the constant zero temperature at the endpoint.

Example 12.1. Solve the IBVP (12.1) with the initial data $\phi(x) = e^x$.

Using the solution formula (12.6), we have

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[e^{-(x-y)^2/4kt}e^y - e^{-(x+y)^2/4kt}e^y\right] dy. \tag{12.7}$$

Combining the exponential factors of the first product under the integral, we will get an exponential with the following exponent

$$\frac{-[y^2 - 2(x+2kt)y + x^2]}{4kt} = -\left(\frac{y-(x+2kt)}{\sqrt{4kt}}\right)^2 + kt - x = -p^2 + kt + x,$$

where we made the obvious notation

$$p = \frac{y - x - 2kt}{\sqrt{4kt}}.$$

Similarly, the exponent of the combined exponential from the second product under integral (12.7) is

$$\frac{-[y^2 + 2(x-2kt)y + x^2]}{4kt} = -\left(\frac{y+x-2kt}{\sqrt{4kt}}\right)^2 + kt - x = -q^2 + kt - x,$$

with

$$q = \frac{y + x - 2kt}{\sqrt{4kt}}.$$

Braking integral (12.7) into a difference of two integrals, and making the changes of variables $y \mapsto p$, and $y \mapsto q$ in the respective integrals, we will get

$$v(x,t) = e^{kt+x} \frac{1}{\sqrt{\pi}} \int_{\frac{-x-2kt}{\sqrt{4kt}}}^{\infty} e^{-p^2}\, dp - e^{kt-x} \frac{1}{\sqrt{\pi}} \int_{\frac{x-2kt}{\sqrt{4kt}}}^{\infty} e^{-q^2}\, dq. \tag{12.8}$$

Notice that

$$\frac{1}{\sqrt{\pi}} \int_{\frac{-x-2kt}{\sqrt{4kt}}}^{\infty} e^{-p^2}\, dp = \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-p^2}\, dp + \frac{1}{\sqrt{\pi}} \int_{\frac{-x-2kt}{\sqrt{4kt}}}^{0} e^{-p^2}\, dp = \frac{1}{2} + \frac{1}{2}\mathcal{E}\mathrm{rf}\left(\frac{x+2kt}{\sqrt{4kt}}\right),$$

and similarly for the second integral. Putting this back into (12.8), we will arrive at the solution

$$v(x,t) = e^{kt+x}\left[\frac{1}{2} + \frac{1}{2}\mathcal{E}\mathrm{rf}\left(\frac{x+2kt}{\sqrt{4kt}}\right)\right] - e^{kt-x}\left[\frac{1}{2} - \frac{1}{2}\mathcal{E}\mathrm{rf}\left(\frac{x-2kt}{\sqrt{4kt}}\right)\right].$$

□

12.1 Neumann boundary condition

Let us now turn to the Neumann problem on the half-line,

$$\begin{cases} w_t - kw_{xx} = 0, & 0 < x < \infty, 0 < t < \infty, \\ w(x,0) = \phi(x), & x > 0 \\ w_x(0,t) = 0, & t > 0. \end{cases} \tag{12.9}$$

To find the solution of (12.9), we employ a similar idea used in the case of the Dirichlet problem. That is, we seek to reduce the IBVP to an IVP on the whole line by extending the initial data $\phi(x)$ to the negative half-axis in such a fashion that the boundary condition is automatically satisfied.

Notice that if $\psi(s)$ is an even function, i.e. $\psi(-x) = \psi(x)$, then its derivative function will be odd. Indeed, differentiating in the definition of the even function, we get $-\psi'(-x) = \psi'(x)$, which is the same as $\psi'(-x) = -\psi'(x)$. Hence, for an arbitrary even function $\psi(x)$, $\psi'(0) = 0$. It is now clear that extending the initial data so that the resulting function is even will produce solutions to the IVP on the whole line that automatically satisfy the Neumann condition of (12.9).

We define the even extension of $\phi(x)$,

$$\phi_{\text{even}} = \begin{cases} \phi(x) & \text{for } x \geq 0, \\ \phi(-x) & \text{for } x \leq 0, \end{cases} \tag{12.10}$$

and consider the following IVP on the whole line

$$\begin{cases} u_t - ku_{xx} = 0, & -\infty < x < \infty, 0 < t < \infty, \\ u(x,0) = \phi_{\text{even}}(x). \end{cases} \tag{12.11}$$

It is clear that the solution $u(x,t)$ of the IVP (12.11) will be even in x, since the difference $[u(-x,t) - u(x,t)]$ solves the heat equation and has zero initial data. We then use the solution formula for the IVP on the whole line to write

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi_{\text{even}}(y)\, dy, \qquad t > 0, \tag{12.12}$$

and take

$$w(x,t) = u(x,t)\big|_{x \geq 0},$$

similar to the case of the Dirichlet problem. One can show that this $w(x,t)$ solves the IBVP (12.9), and use the expression for the heat kernel, as well as the definition (12.10), to write the solution formula as follows

$$w(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt} \right] \phi(y)\, dy. \tag{12.13}$$

Notice that the formulas (12.6) and (12.13) differ only by the sign between the two exponential terms inside the integral.

In terms of heat conduction, the Neumann condition in (12.9) means that there is no heat exchange between the rod and the environment (recall that the heat flux is proportional to the spatial derivative of the temperature). The physical interpretation of formula (12.13) is that the integrand is the contribution of $\phi(y)$ plus an additional contribution, which comes from the lack of heat transfer to the points of the rod with negative coordinates.

Example 12.2. Solve the IBVP (12.9) with the initial data $\phi(x) \equiv 1$.

Using the formula (12.13), we can write the solution as

$$\begin{aligned} w(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt} \right] dy \\ &= \frac{1}{\sqrt{\pi}} \int_{\frac{-x}{\sqrt{4kt}}}^\infty e^{-p^2}\, dp + \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^\infty e^{-q^2}\, dq, \end{aligned}$$

where we made the changes of variables

$$p = \frac{y-x}{\sqrt{4kt}}, \qquad \text{and} \qquad q = \frac{y+x}{\sqrt{4kt}}.$$

Using the same idea as in the previous example, we can write the solution in terms of the $\mathcal{E}$rf function as follows

$$w(x,t) = \left[\frac{1}{2} + \frac{1}{2}\mathcal{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}} \right) \right] + \left[\frac{1}{2} - \frac{1}{2}\mathcal{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}} \right) \right] \equiv 1.$$

So the solution is identically 1, which is clear if one thinks in terms of heat conduction. Indeed, problem (12.9) describes the temperature dynamics with identically 1 initial temperature, and no heat loss at the endpoint. Obviously there is no heat transfer between points of equal temperature, so the temperatures remain steady along the entire rod. □.

12.2 Conclusion

We derived the solution to the heat equation on the half-line by reducing the initial/boundary value problem to the initial value problem over the whole line through appropriate extension of the initial data. In the case of zero Dirichlet boundary condition the odd extension of the initial data automatically guarantees that the solution will satisfy the boundary condition. While for the case of zero Neumann boundary condition the appropriate choice is the even extension. This reflection method relies on the fact that the solution to the heat equation on the whole line with odd initial data is odd, while the solution with even initial data is even.

So far we considered homogeneous wave and heat equations and the associated initial value problems on the whole line, as well as the boundary value problems on the half-line and the finite line (for wave only). The next step is to extend our study to the inhomogeneous problems, where an external heat source, in the case of heat conduction in a rod, or an external force, in the case of vibrations of a string, are also accounted for. We first consider the inhomogeneous heat equation on the whole line,

$$\begin{cases} u_t - k u_{xx} = f(x,t), & -\infty < x < \infty, t > 0, \\ u(x,0) = \phi(x), \end{cases} \tag{15.1}$$

where $f(x,t)$ and $\phi(x)$ are arbitrary given functions. The right hand side of the equation, $f(x,t)$ is called the *source* term, and measures the physical effect of an external heat source. It has units of heat flux (left hand side of the equation has the units of u_t, i.e. change in temperature per unit time), thus it gives the instantaneous temperature change due to an external heat source.

From the superposition principle, we know that the solution of the inhomogeneous equation can be written as the sum of the solution of the homogeneous equation, and a particular solution of the inhomogeneous equation. We can thus break problem (15.1) into the following two problems

$$\begin{cases} u^h_t - k u^h_{xx} = 0, \\ u^h(x,0) = \phi(x), \end{cases} \tag{15.2}$$

and

$$\begin{cases} u^p_t - k u^p_{xx} = f(x,t), \\ u^p(x,0) = 0. \end{cases} \tag{15.3}$$

Obviously, $u = u^h + u^p$ will solve the original problem (15.1).

Notice that we solve for the general solution of the homogeneous equation with arbitrary initial data in (15.2), while in the second problem (15.3) we solve for a particular solution of the inhomogeneous equation, namely the solution with zero initial data. This reduction of the original problem to two simpler problems (homogeneous, and inhomogeneous with zero data) using the superposition principle is a standard practice in the theory of linear PDEs.

We have solved problem (15.2) before, and arrived at the solution

$$u^h(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy, \tag{15.4}$$

where $S(x,t)$ is the heat kernel. Notice that the physical meaning of expression (15.4) is that the heat kernel averages out the initial temperature distribution along the entire rod.

Since $f(x,t)$ plays the role of an external heat source, it is clear that this heat contribution must be averaged out, too. But in this case one needs to average not only over the entire rod, but over time as well, since the heat contribution at an earlier time will effect the temperatures at all later times. We claim that the solution to (15.3) is given by

$$u^p(x,t) = \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) f(y,s)\,dy\,ds. \tag{15.5}$$

Notice that the time integration is only over the interval $[0,t]$, since the heat contribution at later times can not effect the temperature at time t. Combining (15.4) and (15.5) we obtain the following solution to the IVP (15.1)

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) f(y,s)\,dy\,ds, \tag{15.6}$$

or, substituting the expression of the heat kernel,

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y)\, dy + \int_0^t \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/4k(t-s)}}{\sqrt{4\pi k(t-s)}} f(y,s)\, dy\, ds.$$

One can draw parallels between formula (15.6) and the solution to the inhomogeneous ODE analogous to the heat equation. Indeed, consider the IVP for the following ODE.

$$\begin{cases} \frac{d}{dt}u(t) - Au(t) = f(t), \\ u(0) = \phi, \end{cases} \tag{15.7}$$

where A is a constant (more generally, for vector valued u, the equation will be a system of ODEs for the components of u, and A will be a matrix with constant entries). Using an integrating factor e^{-At}, the ODE in (15.7) yields

$$\frac{d}{dt}\left(e^{-At}u\right) = e^{-At}\frac{du}{dt} - Ae^{-At}u = e^{-At}(u' - Au) = e^{-At}f(t).$$

But then

$$e^{-At}u = \int_0^t e^{-As} f(s)\, ds + e^{-A\cdot 0}u(0),$$

and multiplying both sides by e^{At} gives

$$u(t) = e^{At}\phi + \int_0^t e^{A(t-s)} f(s)\, ds. \tag{15.8}$$

The operator $\mathscr{S}(t)\phi = e^{At}\phi$, called the *propagator* operator, maps the initial value ϕ to the solution of the homogeneous equation at later times. In terms of this operator, we can rewrite solution (15.8) as

$$u(t) = \mathscr{S}(t)\phi + \int_0^t \mathscr{S}(t-s) f(s)\, ds. \tag{15.9}$$

In the case of the heat equation, the heat propagator operator is

$$\mathscr{S}(t)\phi = \int_{-\infty}^{\infty} S(x-y,t)\phi(t)\, dy,$$

which again maps the initial data ϕ to the solution of the homogeneous equation at later times. Using the heat propagator, we can rewrite formula (15.6) in exactly the same form as (15.9).

We now show that (15.6) indeed solves the problem (15.1) by a direct substitution. Since we have solved the homogeneous equation before, it suffices to show that u^p given by (15.5) solves problem (15.3). Differentiating (15.5) with respect to t gives

$$\partial_t u^p = \int_{-\infty}^{\infty} S(x-y,0) f(y,t)\, dy + \int_0^t \int_{-\infty}^{\infty} \frac{\partial}{\partial t} S(x-y,t-s) f(y,s)\, dy\, ds.$$

Recall that the heat kernel solves the heat equation and has the Dirac delta function as its initial data, i.e. $S_t = kS_{xx}$, and $S(x-y,0) = \delta(x-y)$. Hence,

$$\begin{aligned} \partial_t u^p &= \int_{-\infty}^{\infty} \delta(x-y) f(y,t)\, dy + \int_0^t \int_{-\infty}^{\infty} k\frac{\partial^2}{\partial x^2} S(x-y,t-s) f(y,s)\, dy\, ds \\ &= f(x,t)\, dy + k\frac{\partial^2}{\partial x^2} \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) f(y,s)\, dy\, ds = f(x,t) + ku^p_{xx}, \end{aligned}$$

which shows that u^p solves the inhomogeneous heat equation. It is also clear that

$$\lim_{t\to 0} u^p(x,t) = \lim_{t\to 0} \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) f(y,s)\, dy\, ds = 0.$$

Thus, u^p given by (15.5) indeed solves problem (15.3), which finishes the proof that (15.6) solves the original IVP (15.1).

Example 15.1. Find the solution of the inhomogeneous heat equation with the source $f(x,t) = \delta(x-2)\delta(t-1)$ and zero initial data.

Using formula (15.6), and substituting the expression for $f(x,t)$, and $\phi(x) = 0$, we get

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)\delta(y-2)\delta(s-1)\, dy\, ds = \int_0^t S(x-2,t-s)\delta(s-1)\, ds.$$

For the last integral, notice that if $t < 1$, then $\delta(s-1) = 0$ for all $s \in [0,t]$, and if $t > 1$, then the delta function will act on the heat kernel by assigning its value at $s = 1$. Hence,

$$u(x,t) = \begin{cases} 0 & \text{for } 0 < t < 1, \\ S(x-2,t-1) & \text{for } t > 1. \end{cases}$$

This, of course, coincides with our intuition of heat conduction, since the external heat source in this case gives an instantaneous temperature boost to the point $x = 1$ at time $t = 1$. Henceforth, the temperature in the rod will remain zero till the time $t = 1$, and afterward the heat will transfer exactly as in the case of the homogeneous heat equation with data given at time $t = 1$ as $u(x,1) = \delta(x-2)$. □

15.1 Source on the half-line

We will use the reflection method to solve the inhomogeneous heat equation on the half-line. Consider the Dirichlet heat problem

$$\begin{cases} v_t - kv_{xx} = f(x,t), & \text{for } 0 < x < \infty, \\ v(x,0) = \phi(x), \\ v(0,t) = h(t). \end{cases} \tag{15.10}$$

Notice that in the above problem not only the equation is inhomogeneous, but the boundary data is given by an arbitrary function $h(t)$. In this case the Dirichlet condition is called inhomogeneous. We can reduce the above problem to one with zero initial data by the following subtraction method. Defining the new quantity

$$V(x,t) = v(x,t) - h(t), \tag{15.11}$$

we have that

$$\begin{aligned} &V_t - kV_{xx} = v_t - h'(t) - kv_{xx} = f(x,t) - h'(t), \\ &V(x,0) = v(x,0) - h(0) = \phi(x) - h(0), \\ &V(0,t) = v(0,t) - h(t) = h(t) - h(t) = 0. \end{aligned}$$

Thus, $v(x,t)$ solves problem (15.10) if and only if $V(x,t)$ solves the Dirichlet problem

$$\begin{cases} V_t - kV_{xx} = f(x,t) - h'(t), & \text{for } 0 < x < \infty, \\ V(x,0) = \phi(x) - h(0), \\ V(0,t) = 0. \end{cases} \tag{15.12}$$

With this procedure, we essentially combined the heat source given as the boundary data at the endpoint $x = 0$ with the external heat source $f(x,t)$. Notice that $h(t)$ has units of temperature, so its derivative will have units of heat flux, which matches the units of $f(x,t)$. We will denote the combined source in the last problem by $F(x,t) = f(x,t) - h'(t)$, and the initial data by $\Phi(x) = \phi(x) - h(0)$. Since the Dirichlet boundary condition for V is homogeneous, we can extend $F(x,t)$ and $\Phi(x,t)$ to the whole line in an odd fashion, and use the reflection method to solve (15.12). The extensions are

$$\Phi_{\text{odd}}(x) = \begin{cases} \phi(x) - h(0) & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -\phi(-x) + h(0) & \text{for } x < 0, \end{cases} \qquad F_{\text{odd}}(x,t) = \begin{cases} f(x,t) - h'(t) & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -f(-x,t) + h'(t) & \text{for } x < 0. \end{cases}$$

Clearly, the solution to the problem

$$\begin{cases} U_t - kU_{xx} = F_{\text{odd}}(x,t), & \text{for } -\infty < x < \infty, \\ U(x,0) = \Phi_{\text{odd}}(x), \end{cases}$$

is odd, since $U(x,t) + U(-x,t)$ will solve the homogeneous heat equation with zero initial data. Then $U(0,t) = 0$, and the restriction to $x \geq 0$ will solve the Dirichlet problem (15.12) on the half-line. Thus, for $x > 0$,

$$V(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\Phi_{\text{odd}}(y)\,dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)F_{\text{odd}}(y,s)\,dy\,ds.$$

Proceeding exactly as in the case of the (homogeneous) heat equation on the half-line, we will get

$$\begin{aligned} V(x,t) = &\int_0^{\infty} [S(x-y,t) - S(x+y,t)](\phi(y) - h(0))\,dy \\ &+ \int_0^t \int_0^{\infty} [S(x-y,t-s) - S(x+y,t-s)](f(y,s) - h'(s))\,dy\,ds. \end{aligned}$$

Finally, using that $v(x,t) = V(x,t) + h(t)$, we have

$$\begin{aligned} v(x,t) = h(t) + &\int_0^{\infty} [S(x-y,t) - S(x+y,t)](\phi(y) - h(0))\,dy \\ &+ \int_0^t \int_0^{\infty} [S(x-y,t-s) - S(x+y,t-s)](f(y,s) - h'(s))\,dy\,ds. \end{aligned}$$

15.2 Conclusion

Using our intuition of heat conduction as an averaging process with the weight given by the heat kernel, we guessed formula (15.6) for the solution of the inhomogeneous heat equation, treating the inhomogeneity as an external heat source. Employing the propagator operator, this formula coincided exactly with the solution formula for the analogous inhomogeneous ODE, which further hinted at the correctness of the formula. However, to obtain a rigorous proof that formula (15.6) indeed gives the unique solution, we verified that the function given by the formula satisfies the equation and the initial condition by a direct substitution. One can then use this formula along with the reflection method to also find the solution for the inhomogeneous heat equation on the half-line.