MATH 425
LECTURE 6

: The Heat Equation (I)
(Maximum Principle)

We will study the one-dimensional heat equation.
As we did with the wave equation, we want to understand the following issues:

1) Explicit solution.
2) Uniqueness
3) Stability of solutions

} ≡ Well-posedness of the problem (*)

- Point 1) is hard for the heat equation. The methods we know so far don't work. We will find the solution by studying first certain properties that the solutions have to satisfy.
- We will show points 2) and 3) by ~~two~~ two different methods: → The Energy Method (as in the wave equation)
  → The Maximum Principle

(*) Well-posedness only requires the uniqueness existence of the solution, not having an explicit formula for it.

• The Energy method: uniqueness

We are going to prove uniqueness of the following initial boundary value problem (IBVP):

$$[I]\begin{cases} u_t - k\,u_{xx} = f(x,t), & x\in[0,L],\ t>0 \\ u(0,t) = g(t),\ u(L,t) = h(t), & \\ u(x,0) = \phi(x), & \end{cases}$$

where f, g, h, ϕ are given functions.

The method is called "energy method" because one finds a ~~positive~~ non negative quantity ("the energy") that is non increasing (in the wave equation, it was preserved in time).

Since in principle we don't need to know what this "energy" is, let's proceed mathematically.

• <u>Uniqueness</u>: (IBVP) [I] has at most one solution.

Proof: By contradiction. Assume $u_1(x,t) \neq u_2(x,t)$ are two solutions, and let $v(x,t) = u_1(x,t) - u_2(x,t)$.

Then, it is easy to check that v satisfies

$$\begin{cases} v_t - k v_{xx} = 0, & 0 \leq x \leq L \\ v(x,0) = 0, & t \geq 0 \\ v(0,t) = v(L,t) = 0. \end{cases}$$

Now, multiply by v:

$$v v_t - k v v_{xx} = 0 \Rightarrow \frac{1}{2}\frac{d}{dt}(v^2) - k v v_{xx} = 0.$$

Integration over $0 < x < L$ followed by integration by parts gives that

$$\frac{1}{2}\frac{d}{dt}\int_0^L (v(x,t))^2 dx - k\int_0^L v(x,t) v_{xx}(x,t) dx =$$

$$= \frac{1}{2}\frac{d}{dt}\int_0^L (v(x,t))^2 dx - \underbrace{k \cancel{v(x,t)} v_x(x,t)\Big]_{x=0}^{x=L}}_{=0} + k\int_0^L (v_x(x,t))^2 = 0 \Rightarrow$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\int_0^L (v(x,t))^2 dx = -k\int_0^L (v_x(x,t))^2 \leq 0 \Rightarrow$$

$$\Rightarrow \int_0^L (v(x,t))^2 dx \leq \int_0^L (v(x,0))^2 dx = 0 \quad \forall t > 0 \Rightarrow$$

$$\Rightarrow \int_0^L (v(x,t))^2 dx = 0 \quad \forall t > 0 \Rightarrow v \equiv 0 \Rightarrow u_1 \equiv u_2 \quad \not\nearrow\!\!\!\swarrow.$$

Remark: Here, the (non negative non increasing) quantity that we called "energy" is $\int_0^L (u(x,t))^2 dx$.

[(*) Non increasing if there are no external sources: $f \equiv 0, \ldots$]

Remark: The result also holds in the whole line case, $-\infty < x < \infty$, if one assumes certain decay conditions (basically, that all the integrals are convergent).

2.2 The Maximum Principle: uniqueness

(Weak) Maximum Principle: If $u(x,t)$ satisfies the heat equation ($u_t = k u_{xx}$) in a rectangle in space-time, $R = \{0 \le x \le L, 0 \le t \le T\}$, then the maximum of $u(x,t)$ is attained either initially ($t=0$) or on the lateral sides ($x=0$ or $x=L$).

- In other words, if we denote $\Gamma = \{(x,t) \in R : t=0 \text{ or } x=0 \text{ or } x=L\}$, then

Weak Maximum Principles: $\max_{(x,t)\in R} \{u(x,t)\} = \max_{(x,t)\in \Gamma} \{u(x,t)\}$

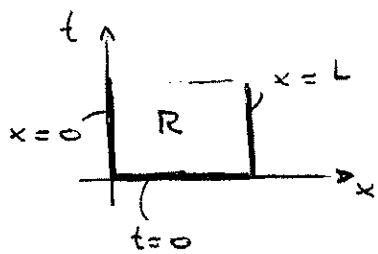
There is a stronger version of this result:

Strong Maximum Principle: $\max_{(x,t)\in R\setminus T} \{u(x,t)\} < \max_{(x,t)\in R} \{u(x,t)\}$.

That is, the (weak) maximum principle ensures that the maximum value of u happens at $t=0$ or at $x=0$ or at $x=L$ (But maybe this maximum could be repeated in some other point).
The maximum principle (strong) ensures that the maximum value of u is <u>only</u> attained at $t=0$ or $x=0$ or $x=L$.

→ We will only proved the weak version.

• <u>Proof</u>: (weak max. principle)



Idea: At an interior maximum (x_0, t_0)

→ 1st derivatives are zero: $u_t(x_0,t_0) = 0$

→ 2nd derivatives are nonpositive: $u_{xx}(x_0,t_0) \le 0$.

~~Bad~~

If we knew that $u_{xx}(x_0,t_0) < 0$, we would have a contradiction: $\underbrace{u_t(x_0,t_0)}_{=0} = k\underbrace{u_{xx}(x_0,t_0)}_{<0}$.

Proof: Denote $M = \max_{(x,t)\in\Gamma} \{u(x,t)\}$

- Goal: To show that $u(x,t) \le M$ for all $(x,t) \in R$.

Steps:

1) Define $v(x,t) = u(x,t) + \varepsilon x^2$, $\varepsilon > 0$.

2) Show that $v(x,t) \le M + \varepsilon L^2$ for all $(x,t) \in R$.

3) Show that 2) implies that $u(x,t) \le M$ for all $(x,t) \in R$.

Let's prove 3) first:

- Proof of 3) $\left[\text{i.e., } v(x,t) \le M + \varepsilon L^2 \Rightarrow u(x,t) \le M\right]$

Since $v(x,t) = u(x,t) + \varepsilon x^2$, we can write

$$\left.\begin{array}{l} u(x,t) = v(x,t) - \varepsilon x^2. \\ \text{Using that } v(x,t) \le M + \varepsilon L^2 \end{array}\right\} \Rightarrow u(x,t) \le M + \varepsilon \underbrace{(L^2 - x^2)}_{\ge 0}.$$

Now, since $\varepsilon > 0$ is arbitrary, this implies (*) that $u(x,t) \le M$ for all $(x,t) \in R$.

~~Let's show this: [illegible] b? ([illegible])~~

~~If $\varepsilon = 0$ trivial. So assume $\varepsilon > 0$.~~

~~Now, by contradiction, assume [illegible].~~

(*) Let's proof this:

$a \le b + \varepsilon c$ for any $\varepsilon > 0 \Rightarrow a \le b$
$(c \ge 0)$ (if $c = 0$ the result is trivial).

How? By contradiction: Assume $a > b$.
Then, we can pick $\varepsilon = \frac{a-b}{2c} > 0$, so by our hypothesis we find that $a \le b + \frac{a-b}{2\cancel{c}}\cancel{c} = \frac{b}{2} + \frac{a}{2} \Rightarrow a \le b$ ↯.

• So we are left to prove point 2).

First, notice that:

2.1) → $v(x,t) \le M + \varepsilon L^2$ for $(x,t) \in \Gamma$.

2.2) → $v_t - K v_{xx} < 0$ $\quad \left[v_t - K v_{xx} = \underbrace{u_t - K u_{xx}}_{=0} - 2\varepsilon K < 0 \right]$

By 2.1), we only need to prove that v attains its maximum on Γ (that is, for $t = 0$ or $x = 0$ or $x = L$).
So let's prove that v cannot reach its maximum value in the interior of R or on the top boundary $t = T$.
(that would prove that the maximum happens on Γ, as all continuous functions reach their maximum in a closed bounded set, like here R).

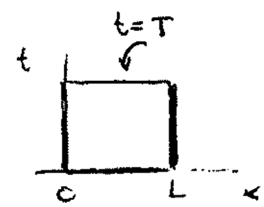
- Suppose $v(x,t)$ attains its maximum at an interior point (x_0,t_0) (that is, $0<x_0<L$, $0<t_0<T$).

Then,

$$\left.\begin{array}{l} v_t = 0 \\ v_{xx} \le 0 \end{array}\right\} \text{at } (x_0,t_0) \rightsquigarrow \text{But then } v_t - k v_{xx} \ge 0 \text{, which contradicts 2.2)!}$$

So v cannot reach its maximum in the interior of R.

- Finally, suppose that $v(x,t)$ has a maximum at a point on the top boundary $t_0=T$, $0<x_0<L$.



Then,

$$\left.\begin{array}{l} v_x(x_0,t_0) = 0 \\ v_{xx}(x_0,t_0) \le 0 \end{array}\right\}$$

and moreover,

$$v_t(x_0,t_0) = \lim_{h\to 0^+} \frac{v(x_0,t_0)-v(x_0,t_0-h)}{h} \ge 0$$ because we are assuming that (x_0,t_0) is a maximum.

All this produces again a contradiction to 2.2)!

→ So, we have concluded that $v(x,t)$ has its maximum in T'. This joined with 2.1) proves that

$$v(x,t) \le \max_{(x,t)\in T'} \{v(x,t)\} \le M + \varepsilon L^2 \text{ for all } (x,t)\in R \quad //.$$

(this proves 2), which was the only thing left//).

We can now use the maximum principle to prove uniqueness of the solution to the heat equation in a different way (recall that we proved uniqueness with the energy method).

<u>Uniqueness</u>: There is at most one solution to the IVBP given by

$$\begin{cases} u_t - K u_{xx} = f(x,t), & 0 \le x \le L \\ u(x,0) = \phi(x), & t > 0 \\ u(0,t) = g(t),\ u(L,t) = h(t) \end{cases}$$

where f, g, h, ϕ are given functions

<u>Proof</u>: As usual, define $v = u_1 - u_2$ with u_1, u_2 two distinct solutions. Then,

$$\begin{cases} v_t - K v_{xx} = 0, & 0 \le x \le L \\ v(x,0) = 0, & t > 0 \\ v(0,t) = v(L,t) = 0. \end{cases}$$

The maximum principle ensures that v attains its maximum and (*) minimum values on $t=0$ or $x=0$ or $x=L$. Since $v=0$ on all those three sides, we conclude that

$$0 \le v(x,t) \le 0 \quad \text{for all } \begin{matrix} x \in [0,L] \\ t \ge 0 \end{matrix} \Rightarrow v \equiv 0 \Rightarrow u_1 \equiv u_2$$ //.

[*] <u>Exercise</u>: Show the "minimum principle": u attains its minimum value at $x=0$ or $x=L$ or $t=0$

2.3| Stability

• Using the maximum principle:

$$\left.\begin{aligned} u_1(x,0) &= \phi_1(x) \\ u_1(0,t) &= g_1(t) \\ u_1(L,t) &= h_1(t) \end{aligned}\right\}(1) \qquad \left.\begin{aligned} u_2(x,0) &= \phi_2(x) \\ u_2(0,t) &= g_2(t) \\ u_2(L,t) &= h_2(t) \end{aligned}\right\}(2)$$

Let u_1, u_2 be two solutions of the heat equation with initial and boundary conditions given by (1), (2), respectively. We want to prove "stability" with respect to the data: that is, if ϕ_1, g_1, h_1 are "similar" to ϕ_2, g_2, h_2, the the solutions u_1, u_2 should also be "similar".

Check that $v = u_1 - u_2$ solves the heat equation with the following conditions:

$$\left.\begin{aligned} &v(x,0) = \phi_1(x) - \phi_2(x) \\ &v(0,t) = g_1(t) - g_2(t), \; v(L,t) = h_1(t) - h_2(t) \end{aligned}\right\}.$$

By the maximum and minimum principle,

$$-\max_{(x,t)\in\Gamma} \{|v(x,t)|\} \le \max_{(x,t)\in R} \{v(x,t)\} \le \max_{(x,t)\in\Gamma} \{|v(x,t)|\}.$$

therefore,

$$\max_{\substack{0\le x\le L\\ 0\le t}} \{|u_1(x,t)-u_2(x,t)|\} \le \max_{(x,t)\in\Gamma} \{|v(x,t)|\} =$$

$$= \max_{\substack{0\le x\le L\\ t\ge 0}} \{|\phi_1(x)-\phi_2(x)|, |g_1(t)-g_2(t)|, |h_1(t)-h_2(t)|\}.$$

- Here we are measuring the "similarity" of functions as the maximum of their difference (in absolute value).

  ↳ Stability in the "uniform sense".

- One can also prove stability, but in the "L^2 sense" (or square integral sense) using the energy method.

↳ <u>Exercise:</u> Let u_1, u_2 be two solutions of $u_t - Ku_{xx} = 0$, with

$$\left.\begin{aligned} u_1(x,0) &= \phi_1(x) \\ u_1(0,t) &= 0 = u_1(L,t) \end{aligned}\right\} \qquad \left.\begin{aligned} u_2(x,0) &= \phi_2(x) \\ u_2(0,t) &= u_2(L,t) = 0 \end{aligned}\right\}.$$

Show, using the energy method, that

$$\int_0^L |u_1(x,t)-u_2(x,t)|^2 dx \le \int_0^L |\phi_1(x)-\phi_2(x)|^2 dx \quad \text{for all } t\ge 0.$$