| MATH 241 LECTURE 5 | : Method of Separation of Variables II.

(~ Sec. 2.4

<u>Last day</u>:

$$\left.\begin{aligned} & u_t = k u_{xx}, \quad 0<x<L,\ t>0 \quad \text{PDE} \\ & \left.\begin{aligned} u(0,t)=0 \\ u(L,t)=0 \end{aligned}\right., \quad t>0 \quad \text{BC} \\ & u(x,0) = f(x), \quad 0<x<L \quad \text{IC} \end{aligned}\right\}$$

Solution using the method of separation of variables:

- $u(x,t) = G(t)\phi(x) \Rightarrow \dfrac{1}{k}\dfrac{G'(t)}{G(t)} = \dfrac{\phi''(x)}{\phi(x)} = -\lambda \longrightarrow G'(t) = -\lambda k G(t)$
  $\searrow \phi''(x) + \lambda\phi(x) = 0.$
- Solve both ODEs for $\lambda>0$, $\lambda<0$, $\lambda=0$.
- Use superposition principle:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

- Lastly impose the initial condition:

[I] $$f(x) = u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right),$$

and find the constants B_n in terms of $f(x)$ using the orthogonality property of $\sin\left(\frac{n\pi x}{L}\right)$:

Orthogonality: $\displaystyle\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & n \neq m \\ L/2 & n = m \end{cases} \quad (n, m = 1, 2, \dots)$.

How to find the B_n?

Multiply [I] by $\sin\left(\frac{m\pi x}{L}\right)$ and integrate from $x=0$ to $x=L$:

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty} B_n \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx =$$

$$= B_m \int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{L}{2}B_m \Rightarrow$$

$$\Rightarrow B_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx.$$

• Thus, the final solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t} \quad \text{with } B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

• [Section 2.4]

2.4.1 Heat conduction in a rod with insulated ends.

$$\left.\begin{aligned} &u_t = k u_{xx}, \quad 0<x<L,\ t>0 && \text{PDE} \\ &\left.\begin{aligned} &u_x(0,t)=0 \\ &u_x(L,t)=0 \end{aligned}\right\}, \ t>0 && \text{BC} \\ &u(x,0)=f(x), \quad 0<x<L && \text{IC} \end{aligned}\right\}$$

Solution: Method of Separation of Variables

(since both the PDE and BC are linear and homogeneous)

Let $u(x,t) = G(t)\phi(x)$. Then, substitution into the PDE gives that

① $G'(t) = -\lambda k G$
② $\phi''(x) = -\lambda\phi(x)$

(same as before).

① $\Rightarrow$ $G(t) = c\,e^{-\lambda k t}$.

② We need to add the BC:

$$\left.\begin{aligned} &u_x(0,t) = 0 = G(t)\phi'(0) \Rightarrow \phi'(0)=0 \\ &u_x(L,t) = 0 = G(t)\phi'(L) \Rightarrow \phi'(L)=0 \end{aligned}\right\}$$ so we need to

Solve the BVP given by

$$\left.\begin{aligned} \phi'' &= -\lambda\phi \\ \phi'(0) &= 0 \\ \phi'(L) &= 0 \end{aligned}\right\}$$ considering the cases $\lambda > 0$, $\lambda = 0$, $\lambda < 0$, (λ complex).

- $\underline{\lambda > 0}$: $\phi(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$

  $\hookrightarrow \phi'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$

  $\phi'(0) = c_2\sqrt{\lambda} = 0 \Rightarrow \underline{\underline{c_2 = 0}}$

  $\phi'(L) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0$ $\nearrow$ $c_1 = 0$ (trivial sol).

  $\hookrightarrow \sin(\sqrt{\lambda}L) = 0 \Leftrightarrow \lambda = \left(\frac{n\pi}{L}\right)^2$

Thus,

$\lambda = \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, \ldots$ are eigenvalues with eigenfunctions

$\phi(x) = c_1 \cos\left(\frac{n\pi x}{L}\right)$.

The corresponding product solutions are

$$u(x,t) = A\cos\left(\frac{n\pi x}{L}\right)e^{-\left(\frac{n\pi}{L}\right)^2 kt}, \quad n = 1, 2, \ldots$$

• $\underline{\lambda = 0}$: $\phi(x) = c_1 + c_2 x$

$$\phi'(x) = c_2 \rightarrow \phi'(0) = \phi'(L) = 0 = c_2 .$$

Thus, $\lambda = 0$ is an eigenvalue with eigenfunction $\phi(x) = 1$, and product solution $u(x,t) = A_0$.

• $\underline{\lambda < 0}$: Check that there are not negative eigenvalues.

• Once we have solved the BVP to find the eigenvalues and eigenfunctions, we use the superposition principle:

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt} =$$

$$= \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt} .$$

• Finally, impose the I.C. to find the A_n:

[I] $f(x) = u(x,0) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$.

We need the following orthogonality property:

· Proposition: Orthogonality of Cosines

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & n \neq m, \\ \frac{L}{2} & n = m \neq 0, \\ L & n = m = 0. \end{cases}$$

With this formula in mind, multiply [I] by $\cos\left(\frac{m\pi x}{L}\right)$ and integrate from $x=0$ to $x=L$:

$$\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx = \sum_{n=0}^{\infty} A_n \int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx$$

If $m=0$: $\displaystyle\int_0^L f(x)\,dx = \sum_{n=0}^{\infty} A_n \int_0^L \cos\left(\frac{n\pi x}{L}\right)dx = L\cdot A_0 \Rightarrow$

$$\Rightarrow A_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

If $m \neq 0$: $\displaystyle\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx = A_m \frac{L}{2} \Rightarrow$

$$\Rightarrow A_m = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx.$$

Remark: Notice that the final solution is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt} \quad \text{with}$$

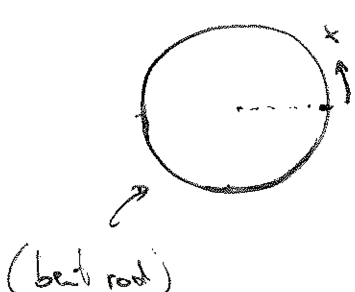
$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx, \quad A_m = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx \quad (m = 1, 2, \ldots).$$

Therefore,

$$\lim_{t\to\infty} u(x,t) = A_0 = \frac{1}{L}\int_0^L f(x)\,dx$$, which coincides (of course)

with the equilibrium solution of the problem.

2.4.1) Heat conduction in an insulated circular ring.



Let's assume length $2L$.

The variable x measures the length from $-L$ to L (for convenience of notation).

PDE: $u_t = k u_{xx}, \quad -L < x < L, \; t > 0$

BC: $u(-L,t) = u(L,t)$, $u_x(-L,t) = u_x(L,t)$, $\quad t > 0$

IC: $u(x,0) = f(x), \quad -L < x < L$

Periodic B.C.

(linear and homogeneous BC.)

<u>Solution:</u>

$$u(x,t) = G(t)\phi(x) \leadsto \begin{cases} G'(t) = -\lambda k G(t) \leadsto G(t) = c e^{-\lambda k t}. \\ \phi'' = -\lambda \phi . \end{cases}$$

<u>BVP</u>: $\phi'' + \lambda\phi = 0$

$$\left.\begin{array}{l} u(-L,t) = G(t)\phi(-L) = u(L,t) = G(t)\phi(L) \leadsto \phi(-L) = \phi(L) \\ u_x(-L,t) = G(t)\phi'(-L) = u_x(-L,t) = G(t)\phi'(-L) \leadsto \phi'(-L) = \phi'(L) \end{array}\right\}$$

That is, $\left|\begin{array}{l} \phi'' + \lambda\phi = 0 \\ \phi(-L) = \phi(L) \\ \phi'(-L) = \phi'(L) \end{array}\right.$

- <u>$\lambda > 0$</u>: $\phi(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$

$$\phi(-L) = \phi(L) \Leftrightarrow c_1 \cancel{\cos}(-\sqrt{\lambda}L) + c_2 \sin(-\sqrt{\lambda}L) = c_1 \cancel{\cos}(\sqrt{\lambda}L) + c_2 \sin(\sqrt{\lambda}L) \Leftrightarrow$$

$$\Leftrightarrow 2c_2 \sin(-\sqrt{\lambda}L) = 0 \Leftrightarrow \lambda = \left(\frac{n\pi}{L}\right)^2, \; n = 1, 2, \ldots$$

$$\phi'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

$$\phi'(L) = \phi'(-L) \Leftrightarrow c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0 \Leftrightarrow \lambda = \left(\frac{n\pi}{L}\right)^2, \; n = 1, 2, \ldots$$

Thus, positive eigenvalues $\lambda = \left(\frac{n\pi}{L}\right)^2$, $n=1,2,\ldots$, with

$$\phi(x) = c_1 \cos\left(\frac{n\pi x}{L}\right) + c_2 \sin\left(\frac{n\pi x}{L}\right), \quad n=1,2,\ldots$$

and correspondingly

$$u(x,t) = \left(c_1 \cos\left(\frac{n\pi x}{L}\right) + c_2 \sin\left(\frac{n\pi x}{L}\right)\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}, \quad n=1,2,\ldots.$$

• $\underline{\lambda=0}$: $\phi(x) = c_1 + c_2 x$

$$\phi(-L) = \phi(L) \iff \cancel{c_1} + c_2 L = \cancel{c_1} - c_2 L \iff c_2 = 0.$$

$$\phi'(-L) = \phi'(L) \iff c_2 = c_2 \checkmark.$$

Thus,

$\lambda = 0$ is eigenvalue with $\phi(x) = c_1$.

• Superposition principle:

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt} + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}.$$

• Initial condition:

[I] $$f(x) = u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right).$$

A_n, B_n?

We need the following property:

• **Proposition:** Orthogonality of Sines and Cosines.

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & n \neq m, \\ L & n = m \neq 0, \\ 2L & n = m = 0, \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & n \neq m. \\ L & n = m \neq 0 \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = 0 .$$

Then, write [I] in the form

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right),$$

multiply by $\cos\left(\frac{m\pi x}{L}\right)$ and integrate from $x=0$ to L to solve for A_m.

Analogously, do the same with $\sin\left(\frac{m\pi x}{L}\right)$ to find B_m //.

One finds that

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx\,, \quad A_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx\,,$$

$$(m = 1, 2, \ldots)\,,$$

$$B_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx\,,$$