HOMEWORK ASSIGNMENT 4, Math 241, Section 002

Name: **Due:** Thursday February 13, 8pm

1. Solve Laplace's equation inside the rectangle $0 \leq x \leq L$, $0 \leq y \leq H$, with the boundary conditions $u_x(0, y) = 0$, $u_x(L, y) = 0$, $u_y(x, H) = 0$, and

$$u(x,0) = \begin{cases} 0, & L/2 < x < L, \\ 1, & 0 < x < L/2. \end{cases}$$

2. Consider $u(x, y)$ satisfying Laplace's equation inside a rectangle $0 < x < L$, $0 < y < H$, subject to the boundary conditions

$$\begin{aligned} u_x(0,y) &= 0, & u_y(x,0) &= 0, \\ u_x(L,y) &= 0, & u_y(x,H) &= f(x). \end{aligned}$$

   a) Without solving the problem, briefly explain the physical condition under which there is a solution to this problem. Hint: Remember that Laplace's equation describes the equilibrium temperature over the region.
   b) Solve the problem by the method of separation of variables. Show that the method works only under the condition of part a).
   c) The solution in part b) has an arbitrary constant. Determine it by consideration of the time-dependent heat equation $u_t = \Delta u = u_{xx} + u_{yy}$ subject to the initial condition $u(x, y, 0) = g(x, y)$. Hint: You might need to use integration by parts in several variables (see end of Lecture 1, or the particular case in formula (1.5.16) on page 25 of Haberman's book).

   Note: For better understanding, you can also read problems 2.5.15.d) and 2.5.16 in the book.

3. Find the solution to the Laplace equation

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

   on the region of the plane outside the disk of radius 1, which is bounded as $r \to +\infty$, and satisfies $u(1, \theta) = 1 + 3\cos(2\theta)$.

4. Solve the Laplace equation inside a semi-infinite strip $0 < x < \infty$, $0 < y < H$, subject to the boundary conditions $u_y(x, 0) = 0$, $u_y(x, H) = 0$, $u(0, y) = f(y)$. Consider only physically reasonable solutions.

   Hint: you may want to use exponentials instead of hyperbolic functions in this problem.

5. a) Determine all non-zero solutions $\phi(x)$ and scalar λ given the ODE

$$x^2\phi''(x) + x\phi'(x) + \lambda^2\phi(x) = 0,$$

for $1 < x < 2$, given the boundary conditions $\phi(1) = 0 = \phi(2)$.
Note: This would appear when solving the Laplace equation in an annulus of radii 1 and 2 with zero temperatures on the inner and outer boundaries.

b) Construct a general solution to the PDE

$$x^2 u_{xx}(x,t) + x u_x(x,t) = u_t(x,t),$$

for $1 < x < 2$ and $t > 0$, given the boundary conditions $u(1,t) = 0$ and $u(2,t) = 0$ for all $t \geq 0$. (Use part a))

6. Consider the function $f : [0,1] \to \mathbb{R}$ given by:

$$f(x) = 1 - x.$$

a) Sketch the graph of the Fourier Sine Series of $f(x)$ for $-3 \leq x \leq 3$. Mark clearly the points of discontinuity, if there are any.

b) Compute the Fourier Sine Series of $f(x)$. Simplify the coefficients.

c) For what values $0 \leq x \leq 1$ does the Fourier Sine Series converge to $f(x)$?

7. Compute the Fourier series of the function $f : [-1,1] \to \mathbb{R}$ given by $f(x) = 1 - |x|$. For what values of $-1 \leq x \leq 1$ does this series converge to $f(x)$?

8. In the Fourier series expansion of $f(x) = x - 1$ on $[-1,1]$,

a) Find the coefficient on the term $\sin(3\pi x)$.

b) At $x = 1$, to what value does the Fourier series converge?

9. Review all the material so far for the first midterm (that is, Lecture Notes 1 to 8. In the book, up to Section 3.3). How much time did this homework assignment take you to complete?