HOMEWORK ASSIGNMENT 2

Name: **Due:** Thursday January 30, 1PM

The level of difficulty is marked as *,**,***.

Problem 1*:

Consider the PDE

$$\frac{1}{x}u_x - yu_y = 0.$$

Find the solution and determine where it is uniquely defined if

a) $u(x,1) = e^{x^2}$,

b) $u(1,y) = e^{y^2}$.

Problem 2**:

Given the equation $xu_x + (x^2+y)u_y + (y/x - x)u = 1$ together with the initial condition $u(1,y) = 0$,

a) Solve the problem for $x > 0$. What is the value of $u(3,6)$?

b) Is the solution defined for all $x > 0$? .

Problem 3***:

Let $\lambda \in \mathbb{R}$. Consider the equation $xu_x + yu_y = \lambda u$.

a) Let $\lambda = 4$. Obtain a solution that satisfies $u(x,y) = 1$ on the circle $x^2 + y^2 = 1$.

b) Let $\lambda = 2$. Find two solutions that satisfy $u(x,0) = x^2$, for every $x > 0$.

Problem 4*:

We want to obtain a more realistic model for a river. Consider that it is defined by the domain

$$R = \{(x,y) : |y| < 1, -\infty < x < \infty\}.$$

Denote by $\rho(x,y,t)$ the concentration of a contaminant at point (x,y) and time t. We assume the velocity of the fluid is only in the x direction (and constant in that direction), but its magnitude depends on y: $u = u(y) = 1 - y^2$ (i.e., it has a parabolic profile, being zero at the boundaries).

a) Write the PDE that describes the evolution of $\rho(x,y,t)$.

b) If initially $\rho(x, y, 0) = e^y e^{-x^2}$, find the concentration ρ for all (x, y, t).

c) With the same initial data, what is the maximum possible concentration at the point $(2, 0)$?

Problem 5*: Strauss, Section 1.3 #2, p.19

A flexible chain of length l is hanging from one end $x = 0$ but oscillates horizontally. Let the x axis point downward and the u axis point to the right. Assume that the force of gravity at each point of the chain equals the weight of the part of the chain below the point and is directed tangentially along the chain. Assume that the oscillations are small. Find the PDE satisfied by the chain.

Problem 6*: Strauss, Section 1.3 #5, p.19

Derive the equation of the one-dimensional diffusion in a medium that is moving along the x axis to the right at constant speed V.

Problem 7*: Strauss, Section 1.3 #9, p.19

This is an exercise on the divergence theorem

$$\iiint_D \nabla \cdot \mathbf{F} d\mathbf{x} = \iint_{\partial D} \mathbf{F} \cdot \mathbf{n} dS,$$

valid for any bounded domain D in space with boundary surface ∂D and unit outward normal vector $\mathbf{n}$. If you never learned it, read Section A.3 (appendix in W. Strauss book). It is crucial that D be bounded. As an exercise, verify it in the following case by calculating both sides separately: $\mathbf{F} = r^2\mathbf{x}$, $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $r^2 = x^2 + y^2 + z^2$, and $D =$ the ball of radius a and center at the origin.

Problem 8*: Strauss, Section 1.3 #10, p.20

If $\mathbf{f}(\mathbf{x})$ is continuous and $|\mathbf{f}(\mathbf{x})| \leq 1/(|\mathbf{x}|^3 + 1)$ for all $\mathbf{x} \in \mathbb{R}^3$, show that

$$\iiint_{\mathbb{R}^3} \nabla \cdot \mathbf{f} d\mathbf{x} = 0.$$

Hint: Take D to be a large ball, apply the divergence theorem on this domain D, and let its radius tend to infinity.

Problem 9:

Read sections 2.1 and 2.2 of W. Strauss book.