MATH 425 LECTURE 2 : The method of characteristics. ($\sim$ Sec. 1.1, 1.2)

A PDE is the analogous to an ODE in several variables. It is an equation that relates the independent variables $x, y, z, \dots$ with a function $u(x, y, z, \dots)$ and its partial derivatives up to order n $(u_x, u_y, u_{xx}, \dots)$.

Ex: $F(x, y, u(x,y), u_x(x,y), u_y(x,y)) = 0$ PDE in two variables of order one.

- We already know how to solve some PDEs:

Examples:

1) $u_{xx}(x,y) = 0 \rightarrow u_x(x,y) = c_1(y) \rightarrow u(x,y) = c_1(y)x + c_2(y)$.

2) $u_{xx}(x,y) + u(x,y) = 0$

$\hookrightarrow u(x,y) = c_1(y)\cos(x) + c_2(y)\sin x$.

3) $u_{xy} + 5u_y = 1$.

Denote $v(x,y) = u_y(x,y)$, so that the equation for v is

$v_x + 5v = 1$.

Therefore,

$$\frac{\partial}{\partial x}\left(e^{5x} v(x,y)\right) = e^{5x} \Rightarrow e^{5x} v(x,y) = \frac{1}{5} e^{5x} + c_1(y) \Rightarrow$$

$$\Rightarrow v(x,y) = \frac{1}{5} + c_1(y) e^{-5x}.$$

Now we can recover $u(x,y)$:

$$u_y(x,y) = \frac{1}{5} + c_1(y) e^{-5x} \Rightarrow u(x,y) = \frac{1}{5} y + c_2(y) e^{-5x} + c_3(x).$$

→ However, most PDEs aren't easy to solve (indeed, few times explicit solutions can be found).

As we did with ODEs, we start studying 1^{st} order PDEs. We limit ourselves to <u>linear</u> PDEs.

[Section 1.2] <u>First-order linear PDEs</u>. (in $\mathbb{R}^2$).

General form: $a(x,y) u_x + b(x,y) u_y + c(x,y) u = f(x,y)$,

where $a(x,y)$, $b(x,y)$, $c(x,y)$, $f(x,y)$ are given functions.

If $f \equiv 0$, the PDE is called homogeneous.

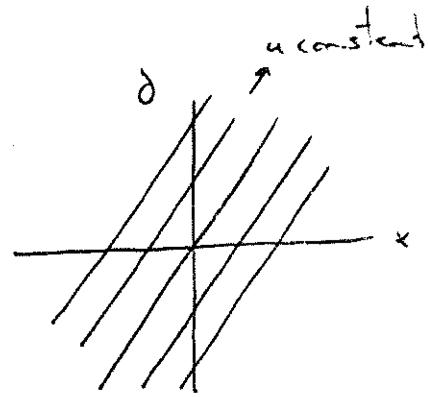
- Example: The constant coefficient case.

PDE $\to$ $3u_x + 4u_y = 0$.

Solution:

Notice that the PDE can be written as $(3,4)\cdot \nabla u = 0$, that is, the directional derivative of $u(x,y)$ in the direction $(3,4)$ is zero.

Thus, u is constant along these lines:



Direction $(3,4)$ $\leadsto$ lines $y = \frac{4}{3}x + c$

$$\left(\frac{dy}{dx} = \frac{4}{3}\right)$$

So the value of $u(x,y)$ only depends on which line we are, that is, on the quantity $c = y - \frac{4}{3}x$.

In conclusion,
$$u(x,y) = f\left(y - \frac{4}{3}x\right) \text{ for any function } f.$$

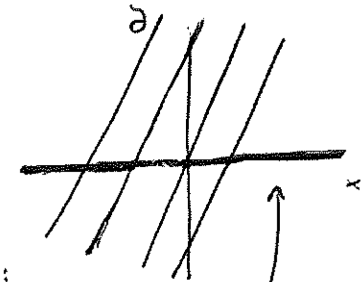
• Ex: Solve $\left.\begin{aligned} 2u_x + 4u_y &= 0 \\ u(x,0) &= \cos(x) \end{aligned}\right\}$

Sol: u is constant along the lines spanned by $(2,4)$, that is,

lines $\frac{dy}{dx} = \frac{4}{2} \to y = 2x + C$.

So $u(x,y) = f(C) = f(y-2x)$.

We can determine f using the data at $y=0$:



$u(x,0) = f(-2x) = \cos(x) \to f(s) = \cos\left(\frac{-s}{2}\right)$.

Thus, $u(x,y) = \cos\left(\frac{2x-y}{2}\right)$ //.

the data is given at $y=0$, which crosses all the characteristic lines.

• Remark: Notice that in both examples we are transforming the PDE into an ODE along the "characteristic lines":

$a u_x + b u_y = 0 \to$ Characteristic lines $y = y(x) = \frac{b}{a}x + C$.

Then,

$$a u_x(x,y(x)) + b u_y(x,y(x)) = a \frac{d}{dx}\Big(u(x,y(x))\Big) = 0 \Rightarrow$$

$$\to u(x,y(x)) = \underbrace{u(x_0, y(x_0)) = u(0,y(0))}_{\text{function that only depends on } C = y - \frac{b}{a}x.} = u(0,C)$$

only determined if we have additional data.

• **Ex:** $\left.\begin{aligned} u_x + u_y + u &= e^{x+2y} \\ u(\cancel{x},0) &= 0 \end{aligned}\right\}$

$u(0,y)$

[(*) With $u(x,0)$ in page -9-*]

Sol: Characteristic lines: $\frac{dy}{dx} = 1 \rightsquigarrow y = x + C$.

We rewrite the equation along these lines $y = y(x)$,

$$u_x(x, y(x)) + u_y(x, y(x)) + u(x, y(x)) = e^{x+2y(x)} \rightsquigarrow$$

$$\rightsquigarrow \frac{d}{dx} v(x) + v(x) = e^{x+2x+2C} = e^{3x} e^{2C}, \quad \text{with } v(x) = u(x, y(x)).$$

So we have the following ODE:

$$\left.\begin{aligned} v'(x) + v(x) &= e^{2C} e^{3x} \\ v(0) &= u(0, y(0)) = 0 \end{aligned}\right\}$$

Using an integrating factor, $\frac{d}{dx}\left(e^x v(x)\right) = e^{2C} e^{4x} \rightsquigarrow$

$$\rightsquigarrow e^x v(x) = e^{2C} \frac{e^{4x}}{4} + c_2 \rightsquigarrow v(x) = e^{2C} \frac{e^{3x}}{4} + c_2 e^{-x}.$$

Since $v(0) = 0 = \frac{1}{4} e^{2C} + c_2$, $c_2 = \frac{-1}{4} e^{2C}$. thus

$$v(x) = \frac{1}{4} e^{2C}\left(e^{3x} - e^{-x}\right).$$

To conclude, recall that $y = x + C \rightsquigarrow C = y - x$.

$$u(x,y) = \frac{1}{4} e^{2y-2x}\left(e^{3x} - e^{-x}\right) = \frac{1}{4}\left(e^{2y+x} - e^{2y-3x}\right)$$

• Example: The variable coefficient case.

$$u_x - y u_y = 0$$

Sol: We follow the previous ideas. We write the PDE as

$$(1,-y) \cdot \nabla u = 0.$$

This means that at each point, the derivative in the direction $(1,-y)$ is zero. But now this direction $(1,-y)$ varies from point to point, so it doesn't define a line. Instead, it defines a curve:

"Characteristic curve": $\frac{dy}{dx} = -y \rightarrow \log(y) = -x + c \rightarrow$

$\rightarrow y = c e^{-x}$ (new constant c).

We can now write the PDE as a ODE along these curves $y = y(x)$,

$$\frac{d}{dx}\left(u(x,y(x))\right) = u_x(x,y(x)) + y'(x) u_y(x,y(x)) = 0 \rightarrow$$

$$\Rightarrow u(x,y(x)) = cte = u(0,y(0)) = u(0,c) = u(0,e^x y) \rightarrow$$

$$\rightarrow u(x,y) = f(e^x y)$$ (for any function f, undetermined)

Question: For the previous PDE $u_x - y u_y = 0$, in which region of the xy plane is the solution uniquely defined if we add the condition $u(x,1) = 2e^x$?

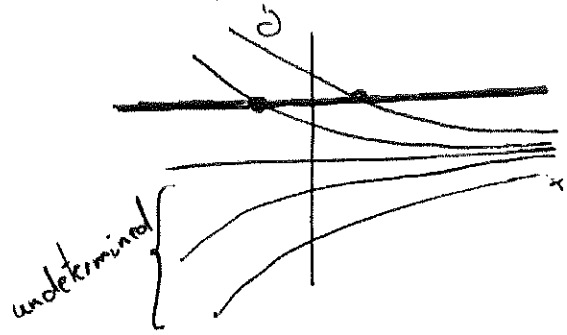
Sol:

We found (using the method of characteristics) that

$u(x,y) = f(e^x y)$.

Now, we impose

$u(x,1) = f(e^x) = 2e^x$



$y = ce^{-x}$
(Charact. curves).

$\Downarrow$

$f(s) = 2s$ for $\underline{s > 0}$ $\rightsquigarrow$ $\| u(x,y) = 2e^x y \|$ uniquely defined for the region $\{(x,y) \in \mathbb{R}^2 : y > 0\}$.

• Remark: We have seen that the method of characteristics consists in converting a PDE into an ODE along certain curves.

In general,

$$a(x,y)\, u_x + b(x,y)\, u_y + c(x,y)\, u = f(x,y),$$

we find the characteristic curves solving the nonlinear equation

$\dfrac{dy}{dx} = \dfrac{b(x,y)}{a(x,y)}$ $\rightsquigarrow$ $y = y(x)$,

which allows us to write the ODE for $u(x, y(x))$.

However, we are assuming there that the characteristic curves can be parametrized by $y = y(x)$, which is an unnecessary restriction. We can instead use a parameter s and find the characteristic curves $(x(s), y(s))$ by solving

$$(x'(s), y'(s)) = (a(x,y), b(x,y)) \rightsquigarrow$$

$$\rightsquigarrow \begin{cases} x'(s) = a(x,y) \\ y'(s) = b(x,y) \end{cases}$$ SDE for the characteristic curves. (generally nonlinear $\to$ <u>hard</u>).

Then,

$$\frac{d}{ds} u(x(s), y(s)) = \underbrace{x'(s)}_{a(x(s),y(s))} u_x(x(s), y(s)) + \underbrace{y'(s)}_{b(x(s),y(s))} u_y(x(s), y(s)).$$

• <u>Example</u>: $\left. \begin{array}{r} x u_x - y u_y + y^2 u = y^2 \\ u(x,x) = 2 \end{array} \right\}$ $\left(\begin{array}{c} \text{also valid} \\ y = \frac{c}{x} \cdots \end{array} \right)$

<u>Sol</u>:

Charact. curves: $\left. \begin{array}{l} x'(s) = x \\ y'(s) = -y \end{array} \right\} \longrightarrow \begin{array}{l} x(t) = c e^{t} \\ y(t) = c e^{-t} \end{array}$

$x(0) = c = y(0)$

Then, $u'(s) + c^2 e^{-2s} u(s) = c^2 e^{-2s}$

$u(0) = u(x(0), y(0)) = u(c, c) = 2$

Integrating factor: $e^{c^2 \int e^{-2s} ds} = e^{-\frac{c^2}{2} e^{-2s}}$ (times any constant).

$$\frac{d}{ds}\left(e^{-\frac{c^2}{2} e^{-2s}} u(s) \right) = c^2 e^{-2s} e^{-\frac{c^2}{2} e^{-2s}} \Rightarrow$$

$$\Rightarrow u(s) = e^{\frac{c^2}{2} e^{-2s}} \left(\int c^2 e^{-2s} e^{\frac{-c^2}{2} e^{-2s}} ds + c_2 \right) =$$

$$= e^{\frac{c^2}{2} e^{-2s}} \left(e^{-\frac{c^2}{2} e^{-2s}} + c_2 \right) = 1 + c_2 e^{\frac{c^2}{2} e^{-2s}}.$$

Using the data $u(0) = 2 = 1 + c_2 e^{\frac{c^2}{2}} \Rightarrow c_2 = e^{-c^2/2}$.

Finally,

$u(x(s), y(s)) = 1 + e^{-\frac{c^2}{2}} e^{\frac{c^2}{2} e^{-2s}}$,

$\left.\begin{array}{l} x(s) = c e^{s} \\ y(s) = c e^{-s} \end{array}\right\} \Rightarrow \begin{array}{l} xy = c^2 \\ y^2 = c^2 e^{-2s} \end{array}$

$\Rightarrow u(x,y) = 1 + e^{-\frac{xy}{2}} e^{y^2/2}$. (defined uniquely everywhere).

- $u_x + u_y + u = e^{x+2y}$
  $u(x,0) = 0$

... we had that $u(x, y(x)) = v(x)$, $y(x) = x + C$,

$v'(x) + v(x) = e^{3x} e^{2C}$ ODE.

I.C. for v?

We only have data for $u(x,0)$.

So let's take the point $x = -C \rightsquigarrow y(-C) = 0$.

$$\left.\begin{aligned} v'(x) + v(x) &= e^{3x} e^{2C} \\ v(-C) = u(-C,0) &= 0 \end{aligned}\right\} \rightarrow$$

$$\rightarrow v(x) = e^{2C} \frac{e^{3x}}{4} + c_2 e^{-x}$$

$$v(-C) = 0 = \frac{e^{2C}}{4} e^{-3C} + c_2 e^{C} \Rightarrow c_2 = -\frac{1}{4} e^{-2C}.$$

Thus,

$$v(x) = \frac{1}{4}\left(e^{2C} e^{3x} - e^{-2C} e^{-x}\right) \xrightarrow{C = y - x}$$

$$\rightarrow u(x,y) = \frac{1}{4}\left(e^{2y-2x} e^{3x} - e^{-2y+2x} e^{-x}\right) = \frac{1}{4}\left(e^{2y+x} - e^{-2y+x}\right)$$