HOMEWORK ASSIGNMENT 3

Name: **Due:** Thursday February 6, 10am

1. Determine how β must be related to L if a time-independent (equilibrium) temperature distribution is to exist for the following PDE

$$u_t(x,t) = u_{xx}(x,t) + x - \beta,$$

for $0 < x < L$ and $t > 0$, with boundary conditions

$$u_x(0,t) = 0, \qquad u_x(L,t) = 0,$$

for $t \geq 0$. Is this equilibrium solution unique for the value of β determined? Explain.

2. Consider the heat equation $u_t = ku_{xx}$ for $0 < x < L$, subject to the boundary conditions $u(0,t) = 0$, $u(L,t) = 0$ for $t \geq 0$. Solve the initial value problem if the temperature is initially

a) $u(x,0) = 6\sin\left(\frac{9\pi x}{L}\right)$,

b) $u(x,0) = \begin{cases} 1, & 0 < x \leq L/2, \\ 2, & L/2 < x < L. \end{cases}$

3. Evaluate

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx, \qquad \text{for } n \geq 0, m \geq 0.$$

Hint: Use the trigonometric identity

$$\cos(a)\cos(b) = \frac{1}{2}\big(\cos(a+b) + \cos(a-b)\big).$$

4. Consider $u_t = ku_{xx} - \alpha u$. Suppose that the boundary conditions are $u(0,t) = u(L,t) = 0$ for $t \geq 0$.

a) What are the possible equilibrium solutions if $\alpha > 0$?

b) Solve the time-dependent problem with $u(x,0) = f(x)$ if $\alpha > 0$. Find the limit as $t \to \infty$ and compare to part a).

5. Consider the PDE $u_x(x,y) = xu_y(x,y)$, for $a < x < b$ and $c < y < d$. If non-zero solutions of the form $u(x,y) = \alpha(x)\beta(y)$ are to be constructed for this PDE, then determine up to three arbitrary constants the forms of $\alpha(x)$ and $\beta(y)$.

6. Decide whether the following statements are true or false:

   a) Consider the heat equation $u_t = u_{xx}$ on a one-dimensional rod of length L subject to the boundary conditions

   $$u(0, t) = 5, \qquad u(L, t) = 8.$$

   The sum of any two solutions (to the given heat equation and boundary conditions) is again a solution.

   b) The method of separation of variables (when it works) solves a PDE by converting it into ordinary differential equations.

7. Solve the heat equation $u_t = 2u_{xx}$, $0 < x < 1$, $t > 0$, subject to $u_x(0, t) = u_x(1, t) = 0$ for $t \geq 0$, in the following cases:

   a) $u(x, 0) = 2 + 7\cos(4\pi x)$,

   b) $u(x, 0) = -2\sin(\pi x)$.

8. Consider the heat equation $u_t = 3u_{xx}$ where $0 < x < 1$ and $t > 0$, with boundary conditions $u(0, t) = 0 = u_x(1, t)$, and initial condition

   $$u(x, 0) = 4\sin\left(\frac{3\pi x}{2}\right) + 7\sin\left(\frac{5\pi x}{2}\right).$$

   a) What is the physical meaning of the boundary condition at the left end-point $x = 0$?

   b) Is there any heat source?

   c) Solve the above initial boundary value problem. You may assume for this problem that no solutions of the heat equation exponentially grow in time. You may also guess appropriate orthogonality conditions for the eigenfunctions.

9. Read Sections (2.5), (3.1), (3.2) and (3.3).