MATH 115

LECTURE 18 :

Last day:

Discrete Random Variable

$X \to R(X) = \{x_1, x_2, \dots\}$

Probability distribution $\to f(x)$,

Cumulative distrib: $F(x_n) = \sum_{k \le n} f(x_k)$,

$\hookrightarrow P(X \le x_n) = F(x_n)$

Remark: $P(X = x_n) = f(x_n)$,

and so $P(X \le x_n) \ne P(X < x_n)$.

Expected Value:

$$E(X) = \sum_k x_k f(x_k), (= \mu)$$

Variance:

$$Var(X) = E((X-\mu)^2) =$$

$$= E(X^2) - \mu^2 =$$

$$= \sum_k x_k^2 f(x_k) - \mu^2$$

Continuous Random Variable

$X \to R(X) = [a, b] \subseteq \mathbb{R}$.

Density distribution $\to f(x)$,

Cumulative distrib.: $F(x) = \int_a^x f(s)\, ds$,

$\hookrightarrow P(X \le x) = P(X < x) = F(x)$.

Remark: $P(X = x) = 0$

Expected Value: $E(X) = \int_a^b x f(x)\, dx$.
(μ)

Variance: $Var(X) = E((X-\mu)^2) =$

$$= E(X^2) - \mu^2 =$$

$$= \int_a^b x^2 f(x)\, dx - \mu^2.$$

!! $\to F(x)$ is discontinuous for X discrete; continuous for X continuous !!

• <u>Exercise</u>: Consider two points x, y taken at random from the segment $[0,1]$. Let the random variable X be the distance between them.

1) Find the density function of X.

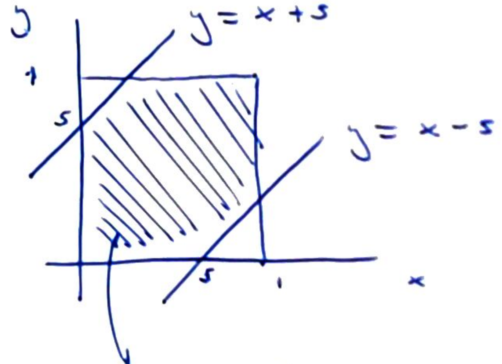
2) Find the probability that the two points x, y are separated by more than $1/2$.

<u>Sol</u>:

$X = |x-y| \to F(s) = P(X < s) = P(|x-y| < s) = \quad \{ s \in [0,1] \}$

$= P(-s < x-y < s)$.

We can represent this graphically $\begin{cases} x-y < s \leftrightarrow y > x-s \\ x-y > -s \leftrightarrow y < x+s \end{cases}$



all these points (x,y) have $|x-y| < s$

$\to P(|x-y| < s) = \dfrac{\#\text{ area}}{\text{total area}} =$

$= \dfrac{1-(1-s)^2}{1} = s(2-s)$,

$(s \in [0,1])$.

Therefore, $f(s) = F'(s) = 2 - 2s = 2(1-s)$.

2) $P(X > \frac{1}{2}) = 1 - P(X \leq \frac{1}{2}) = 1 - \frac{1}{2}(2 - \frac{1}{2}) = 1 - \frac{1}{2}\frac{3}{2} = \frac{1}{4}$.

• Exercise: The cumulative distribution of X is given by

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 - k e^{-\lambda x} & \text{if } x > 0. \end{cases} \qquad (\lambda > 0).$$

a) Value of k?

b) Find the density function of X.

c) Find the expected value of X.

Sol:

a) $\lim_{x \to +\infty} F(x) = \lim_{x \to +\infty} (1 - k e^{-\lambda x}) = 1$ for all k... not useful.

But notice that $F(x)$ must always be continuous, since

$F(x) = P(X \leq x) = P(X < x)$ because X is a continuous r.v.

Therefore,
$$F(0) = 1 - k = 0 \Rightarrow \boxed{k = 1}$$

b) $f(x) = F'(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & x > 0 \end{cases}$

c) $$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} x \lambda e^{-\lambda x}\,dx = \begin{Bmatrix} u = x \to du = dx \\ dv = \lambda e^{-\lambda x} dx \to v = -e^{-\lambda x} \end{Bmatrix}$$

$$= -x e^{-\lambda x}\Big]_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\,dx = -\frac{e^{-\lambda x}}{\lambda}\Big]_0^{\infty} = \frac{1}{\lambda}.$$

• Exercise: $f(x) = \begin{cases} \beta e^{-\beta x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$ $(\beta > 0)$.

1) $E(X)$.

2) $F(x)$.

3) $P(X \geq x + \varepsilon \mid X \geq x)$ for $x > 0, \varepsilon > 0$.

Sol:

1) $E(X) = 1/\beta$.

2) $F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\beta x} & x \geq 0 \end{cases}$

3) $$P(X \geq x + \varepsilon \mid X \geq x) = \frac{P((X \geq x+\varepsilon) \cap (X \geq x))}{P(X \geq x)} = \frac{P(X \geq x+\varepsilon)}{P(X \geq x)} =$$

$$= \frac{1 - P(X < x+\varepsilon)}{1 - P(X < x)} = \frac{1 - F(x+\varepsilon)}{1 - F(x)} = \frac{1 - e^{-\beta x} e^{-\beta \varepsilon}}{1 - e^{-\beta x}}.$$