Lecture 15: Conditional Probability and Independence

Math 115

October 29, 2019

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This second case is an example of **conditional probability**.

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2. If three are chosen, what is the probability that the first is a junior and the next two sophomores?

Problem: A lot contains 12 items, of which 4 are defective. Three items are drawn at random from the lot one after the other. Find the probability that all 3 are non-defective.

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(Remark: Also do this one using a *tree diagram*)

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Exercise: Show that if E and F are independent, the so are E^c and F^c. Also E and F^c.

Exercise: Let E, F, G be three independent events with $P(E) = 5/10$, $P(F) = 4/10$ and $P(G) = 3/1-$. Find $P(E \cap F \cap G)$, $P(E \cap G^c)$, $P(E \cap (F \cup G)^c)$, $P(E \cup (F \cap g)^c)$.

Independent repeated trials

Definition: Let S be a finite probability space. The probability space of n **independent trials**, S_n, consists of ordered n-tuples of elements of S, with probability

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Problem: A machine produces defective items with probability p.

1. If 10 items are chosen at random, what is the probability that exactly 3 are defective?
2. What is the probability of finding at least one defective item in the 10 chosen?
3. If we observe the items one at a time as they come off the line, what is the probability that the third defective item is the tenth item observed?

Example: A city of 100000 people is broken into 4 precincts of unequal size P_1, P_2, P_3, P_4. Their populations are $10000, 20000, 30000, 40000$, respectively. A review of crimes recorded shows that:

- 20% of records in P_1 contain errors.
- 5% of records in P_2 contain errors.
- 10% of records in P_3 contain errors.
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1. Draw a **tree diagram** describing the results.
2. Find the probability that a record has an error and is in P_3.
3. Find the probability that a record has an error.
4. Find the probability that a record is in P_3 given that is has an error.

Problem: A test for a certain allergy tests positive 98% of the time if the person has that allergy, while it only tests positive 1% of the time if the person doesn't have it (false positive). Given that only 3% of the population have this allergy, what is the probability that a patient is allergic if it tests positive?

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Problem: A crate of apples contains 3 bad apples and 7 good ones. Apples are chosen until we pick a good one. What is the probability that it takes at least 3 picks to get a good one?

Bayes' Theorem

Problem: We have two coins. Coin 1 is a fair coin while Coin 2 has two heads. We select a coin randomly and toss it. Say a head comes up.

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Problem: We have two coins. Coin 1 is a fair coin while Coin 2 has two heads. We select a coin randomly and toss it. Say a head comes up.

1. What is the probability that it is Coin 1?
2. Flip the coin again and say a head comes up again. What is the probability that it is Coin 1?