3.34. Prove Corollary 3.7: For any events A, B, C, we have:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Let $D = B \cup C$. Then $A \cap D = A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Using the addition rule (Theorem 3.6), we get

$$\begin{aligned} P(A \cap D) &= P[(A \cap B) \cup (A \cap C)] = P(A \cap B) + P(A \cap C) - P(A \cap B \cap A \cap C) \\ &= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C) \end{aligned}$$

Using the addition rule (Theorem 3.6) again, we get

$$\begin{aligned} P(A \cup B \cup C) &= P(A \cup D) = P(A) + P(D) - P(A \cap D) = P(A) + P(B \cup C) - P(A \cap D) \\ &= P(A) + [P(B) + P(C) - P(B \cap C)] - [P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)] \\ &= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \end{aligned}$$

3.35. A die is tossed 100 times. The following table lists the six numbers and the frequency with which each number appeared:

Number	1	2	3	4	5	6
Frequency	14	17	20	18	15	16

(*a*) Find the relative frequency f of each of the following events:

$$A = \{3 \text{ appears}\},\ B = \{5 \text{ appears}\},\ C = \{\text{even number appears}\}$$

(*b*) Find a probability model of the data.

(*a*) The relative frequency $f = \dfrac{\text{number of successes}}{\text{total number of trials}}$. Thus

$$f_A = \frac{20}{100} = 0.20, \qquad f_B = \frac{15}{100} = 0.15, \qquad f_C = \frac{17 + 18 + 16}{100} = 0.52$$

(*b*) The geometric symmetry of the die indicates that we first assume an equal probability space. Statistics is then used to decide whether or not the given data supports the assumption of a fair die.

Supplementary Problems

SAMPLE SPACES AND EVENTS

3.36. Let A and B be events. Find an expression and exhibit the Venn diagram for the event that:

(*a*) A or not B occurs, (*b*) only A occurs.

3.37. Let A, B, and C be events. Find an expression and exhibit the Venn diagram for the event that:

(*a*) A or C, but not B occurs,
(*b*) exactly one of the three events occurs,
(*c*) none of the events occurs,
(*d*) at least two of the events occur.

3.38. A penny, a dime, and a die are tossed. Describe a suitable sample space S, and find $n(S)$.

3.39. For the space S in Problem 3.38, express explicitly the following events:

$$A = \{\text{two heads and an even number}\}, \qquad B = \{2 \text{ appears}\}$$
$$C = \{\text{exactly one head and an odd number}\}$$

3.40. For the events A, B, C in Problem 3.39, express explicitly the event:

(a) A and B, (b) only B, (c) B and C, (d) A but not B.

FINITE EQUIPROBABLE SPACES

3.41. Determine the probability of each event:

(a) An odd number appears in the toss of a fair die.
(b) 1 or more heads appear in the toss of 4 fair coins.
(c) Both numbers exceed 4 in the toss of 2 fair dice.
(d) Exactly one 6 appears in the toss of 2 fair dice.
(e) A red or a face card appears when a card is randomly selected from a 52-card deck.

3.42. A student is chosen at random to represent a class with 5 freshmen, 4 sophomores, 8 juniors, and 3 seniors. Find the probability that the student is

(a) a sophomore (b) a senior (c) a junior or a senior

3.43. One card is selected at random from 25 cards numbered 1 to 25. Find the probability that the number on the card is: (a) even, (b) divisible by 3, (c) even and divisible by 3, (d) even or divisible by 3, (e) ends in the digit 2.

3.44. Three bolts and three nuts are in a box. Two parts are chosen at random. Find the probability that one is a bolt and one is a nut.

3.45. A box contains 2 white sox, 2 blue sox, and 2 red sox. Two sox are drawn at random. Find the probability they are a match (same color).

3.46. Of 120 students, 60 are studying French, 50 are studying Spanish, and 20 are studying both French and Spanish. A student is chosen at random. Find the probability that the student is studying:

(a) French and Spanish
(b) French or Spanish
(c) neither French nor Spanish
(d) only French
(e) exactly one of the two languages.

3.47. Of 10 girls in a class, 3 have blue eyes. Two of the girls are chosen at random. Find the probability that:

(a) both have blue eyes
(b) neither has blue eyes
(c) at least one has blue eyes
(d) exactly one has blue eyes.

3.48. Ten students A, B, ... are in a class. A committee of 3 is chosen from the class. Find the probability that

(a) A belongs to the committee.
(b) B belongs to the committee.
(c) A and B belong to the committee.
(d) A or B belongs to the committee.

FINITE PROBABILITY SPACES

3.49. Under which of the following functions does $S = \{a_1, a_2, a_3\}$ become a probability space?

(*a*) $P(a_1) = 0.3$, $P(a_2) = 0.4$, $P(a_3) = 0.5$
(*b*) $P(a_1) = 0.7$, $P(a_2) = -0.2$, $P(a_3) = 0.5$
(*c*) $P(a_1) = 0.3$, $P(a_2) = 0.2$, $P(a_3) = 0.5$
(*d*) $P(a_1) = 0.3$, $P(a_2) = 0$, $P(a_3) = 0.7$

3.50. A coin is weighted so that heads is three times as likely to appear as tails. Find $P(H)$ and $P(T)$.

3.51. Suppose A and B are events with $P(A) = 0.7$, $P(B) = 0.5$, and $P(A \cap B) = 0.4$. Find the probability that

(*a*) A does not occur.
(*b*) A or B occurs.
(*c*) A but not B occurs.
(*d*) Neither A nor B occurs.

3.52. Consider the following probability distribution:

Outcome	1	2	3	4	5	6
Probability	0.1	0.3	0.1	0.2	0.2	0.1

Consider the following events:

$$A = \{\text{even number}\}, \qquad B = \{2, 3, 4, 5\}, \qquad C = \{1, 2\}$$

Find: (*a*) $P(A)$, (*b*) $P(B)$, (*c*) $P(C)$, (*d*) $P(\varnothing)$, (*e*) $P(S)$.

3.53. For the events A, B, C in Problem 3.52, find:

(*a*) $P(A \cap B)$, (*b*) $P(A \cup C)$, (*c*) $P(B \cap C)$, (*d*) $P(A^c)$, (*e*) $P(B \cap C^c)$.

3.54. Three students A, B, and C are in a swimming race. A and B have the same probability of winning and each is twice as likely to win as C. Find the probability that

(*a*) B wins (*b*) C wins (*c*) B or C wins

3.55. Let P be a probability function on $S = \{a_1, a_2, a_3\}$. Find $P(a_1)$ if

(*a*) $P(a_2) = 0.3$, $P(a_3) = 0.5$;
(*b*) $P(a_1) = 2P(a_2)$ and $P(a_3) = 0.7$;
(*c*) $P(\{a_2, a_3\}) = 2P(a_1)$;
(*d*) $P(a_3) = 2P(a_2)$ and $P(a_2) = 3P(a_1)$.

ODDS

3.56. Find the probability of an event E if the odds that it will occur are: (*a*) 2 to 1, (*b*) 5 to 11.

3.57. Find the odds that an event E occurs if: (*a*) $P(E) = 2/7$, (*b*) $P(E) = 0.4$.

3.58. In a swimming race, the odds that A will win are 2 to 3 and the odds that B will win are 1 to 4. Find the probability p and the odds that: (*a*) A will lose, (*b*) A or B will win, (*c*) neither A nor B will win.

NONCOUNTABLE UNIFORM SPACES

3.59. A point is chosen at random inside a circle with radius r. Find the probability p that the point is at most $\frac{1}{3}r$ from the center.

3.60. A point A is selected at random inside an equilateral triangle whose side length is 3. Find the probability p that the distance of A from any corner is greater than 1.

3.61. A coin of diameter 1/2 is tossed randomly onto the plane $\mathbf{R}^2$. Find the probability p that the coin does not intersect any line of the form: (*a*) $x = k$ or $y = k$ where k is an integer, (*b*) $x + y = k$ where k is an integer.

3.62. A point X is selected at random from a line segment AB with midpoint O. Find the probability p that the line segments AX, XB, and AO can form a triangle.

MISCELLANEOUS PROBLEMS

3.63. A die is tossed 50 times. The following table gives the 6 numbers and their frequency of occurrence:

Number	1	2	3	4	5	6
Frequency	7	9	8	7	9	10

Find the relative frequency of each event: (*a*) 4 appears, (*b*) an odd number appears, (*c*) a number greater than 4 appears.

3.64. Use mathematical induction to prove: For any events $A_1, A_2, \ldots, A_n$,

$$P(A_1 \cup \cdots \cup A_n) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots \pm P(A_1 \cap \cdots \cap A_n)$$

Remark: This result generalizes Theorem 3.6 (addition rule) for two sets and Corollary 3.7 for three sets.

3.65. Consider the countably infinite sample space $S = \{a_1, a_2, a_3, \ldots\}$. Suppose $P(a_1) = 1/4$ and suppose $P(a_{k+1}) = rP(a_k)$ for $k = 1, 2, \ldots$. Find r and $P(a_3)$.

Answers to Supplementary Problems

3.36. (*a*) $A \cup B^c$; (*b*) $A \cap B^c$.

3.37. (*a*) $(A \cup C) \cap B$; (*b*) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$; (*c*) $(A \cup B \cup B)^c = A^c \cap B^c \cap C^c$; (*d*) $(A \cap B) \cup (A \cap C) \cup (B \cap C)$.

3.38. $n(S) = 24$; $S = \{H, T\} \times \{H, T\} \times \{1, 2, , \ldots, 6\} = \{HH1, \ldots, HH6, HT1, \ldots, TT6\}$.

3.39. $A = \{HH2, HH4, HH6\}$; $B = \{HH2, HT2, TH2, TT2\}$; $C = \{HT1, HT3, HT5, TH1, TH3, TH5\}$.

3.40. (*a*) $\{HH2\}$; (*b*) $\{HT2, TH2, TT2\}$; (*c*) $\varnothing$; (*d*) $\{HH4, HH6\}$.

3.41. (*a*) 3/6; (*b*) 15/16; (*c*) 4/36; (*d*) 10/36; (*e*) 32/52.

3.42. (*a*) 4/20; (*b*) 3/20; (*c*) 11/20.

3.43. (*a*) 12/25; (*b*) 8/25; (*c*) 4/25; (*d*) 16/25; (*e*) 3/25.

3.44. 9/15 = 3/5.

3.45. 3/15 = 1/5.

3.46. (a) 1/6; (b) 3/4; (c) 1/4; (d) 1/3; (e) 7/12.

3.47. (a) 1/15; (b) 7/15; (c) 8/15; (d) 7/15.

3.48. (a) 3/10; (b) 3/10; (c) 1/15; (d) 8/15.

3.49. (c) and (d).

3.50. $P(H) = 3/4$; $P(T) = 1/4$.

3.51. (a) 0.3; (b) 0.8; (c) 0.2; (d) 0.2.

3.52. (a) 0.6; (b) 0.8; (c) 0.4; (d) 0; (e) 1.

3.53. (a) 0.5; (b) 0.7; (c) 0.3; (d) 0.4; (e) 0.5.

3.54. (a) 2/5; (b) 1/5; (c) 3/5.

3.55. (a) 0.2; (b) 0.2; (c) 1/3; (d) 0.1.

3.56. (a) 2/3; (b) 5/16.

3.57. (a) 2 to 5; (b) 2 to 3.

3.58. (a) $p = 3/5$, odds 3 to 2; (b) $p = 3/5$, odds 3 to 2; (c) $p = 2/5$, odds 2 to 3.

3.59. 1/9.

3.60. $1 - 2\pi/(9\sqrt{3}) = 1 - 2\sqrt{3}\,\pi/27$.

3.61. (a) 1/4; (b) $1 - \sqrt{2}/2$.

3.62. 1/2.

3.63. (a) 7/50; (b) 24/50; (c) 19/50.

3.65. $r = 3/4$; $P(a_3) = 9/64$.