Lecture 14: Counting techniques and Probability

Math 115

October 17, 2019

Problem: We consider a club with three members A, B and C.

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2. How many ways can they choose a two person committee?

Permutations and Combinations

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- Lecture 13 $\rightarrow P(n,r) = n(n-1)\cdots(n-(r-1)) = \frac{n!}{(n-r)!}$.
- $C(n,r)$?

In Problem 2, we see that the possibles committees are AB, AC and BC: $3 = \frac{6}{2} = \frac{P(3,2)}{2!}$

AB	BA
AC	CA
BC	CB

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Examples:

1. 8 students try out for quarterback. How many ways can the coach choose a first, second and third string quarterback?
2. 20 Penn students apply for a job with Google. How many ways can Google choose three of them for jobs?

With or without replacement, repeated objects

1. Problem: Flip a coin ten times and record the results:
   1.1 How many possible outcomes are there?
   1.2 How many of these have exactly 5 heads?
   1.3 How many outcomes have at most 3 heads?
   1.4 How many outcomes have at least 2 heads?

   Hint: How many five-letter words can we form with the letters from $BABBY$?

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   Hint: How many five-letter words can we form with the letters from $BABBY$? In general, the number permutations of n objects with n_1, n_2, $\ldots$, n_m repeated objects is

$$P(n; n_1, n_2, \ldots, n_m) = \frac{n!}{n_1! n_2! \cdots n_m!}$$

Sampling with and without replacement

1. Problem: Given ten balls numbered $0, 1, \ldots, 9$, choose k balls with replacement and keep track of the order picked. How many ways can we do this?
2. 6 balls are chosen with replacement from balls labeled $1, 2, 3, \ldots, 30$.
   - 2.1 How many ways can this be done?
   - 2.2 Of these, how many ways has at least one of the numbers $1, 2, \ldots, 10$ show up?

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  1. 0-element sets: $\binom{26}{0}$

Binomial theorem

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  So the answer is

$$\binom{26}{0} + \binom{26}{1} + \cdots + \binom{26}{26} = (1+1)^{26} = 2^{26}$$

Theorem

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n} x^0 y^n.$$

Idea: $(x+y)^n = (x+y)(x+y)\cdots(x+y) \to$ How many ways can obtain $x^r y^{n-r}$ when you multiply?

Some more problems

1. How many ways are there to choose at least two films from a collection of 9?
2. 21 soccer players are to be divided into 3 teams of 7. How many ways can you do this?

Probability

The *Probability* of an *event* represents the long run likelihood that it will happen.
It is always a number p between 0 and 1. Definitions:

- **Experiment**: An activity with an observable outcome.
- **Trial**: Each repetition of the experiment.
- **Outcome**: The result of the trial.
- **Sample space (S)**: The set of all possible outcomes.
- **Event**: A subset of the sample space.

Probability

Some *experiments*:

1. Flip a coin and observe the side that is up.
2. Choose a student and record the student's birthday.
3. Roll two dice and record the sum of the two sides that show on top.
4. Follow a patient after a course of treatment for 5 years and observe the recovery time (in days).
5. Measure and record the height of a subject.

For each experiment:

- Examples of outcomes?
- What is the sample space?
- Example of events?

Probability

Modeling (axioms) of probability:

To every event E we assign a **probability**, $P(E)$, that has to satisfy the following:

1. For every event A, it holds that $P(A) \geq 0$
2. $P(S) = 1$
3. For every sequence of disjoint events $A_1, A_2, \ldots$, we have that

$$P(\cup_{n\geq 1} A_n) = \sum_{n\geq 1} P(A_n).$$

Some consequences are:

- $P(\emptyset) = 0$.
- For every event A, $P(A^c) = 1 - P(A)$.
- If $A \subset B$, then $P(A) \leq P(B)$.
- $0 \leq P(A) \leq 1$ for any event A.
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Example (remember lecture 13): Of the 500 students in some college, 400 are taking a math course, 300 are taking an economics course, and 250 are taking both a math and an economics course. How many are taking neither a math nor an econ course?

Rewrite it as: The probability that any given student at some college is taking a math course is $4/5$. The probability that a student is taking an economics course is $3/5$. The probability that a student is taking both is $1/2$. What is the probability that a student chosen at random is taking neither a math course nor an econ course?

- What is the sample space?
- We want to know $P((M \cup E)^c)$.

Probability

Remark: The previous case is an example of **simple** sample space: There is a finite number of outcomes and they are equally probable. In the simple sample space case, the probability of an event E is given by

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We can however record both dice separately and make it a simple sample space:

Probability

Sample space of rolling two dice:

$$\begin{array}{l}(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\(6,1),(6,2),(6,3),(6,4),(6,5),(6,6),\end{array}$$

So here $\#(S) = 36$ and each outcome is equally probable.

- Determine $P(\text{Sum is } 7)$.

In summary, to use $P(E) = \frac{\#E}{\#S}$, one needs

1. A finite sample space with equally probably outcomes (that is, a simple sample space)
2. The ability to compute $\#E$ and $\#S$.

Probability: problems

1. Worms have invaded 5 apples in a crate of 1000 apples. An inspector checks 10 apples at random for worms. What is the probability that the crate passes?
2. What is the probability that in a group of 10 people at least 2 will share the birthday date? (Ignore leap years) How about a group of 60 people?
3. (Example of **Discrete infinite sample space**) Consider the positive integers $\{n : n = 1, 2, 3, \dots\}$, such that the probability of n happening is $1/2^n$ (note that the sum of the probabilities is 1). What is the probability of choosing a number greater of equal to 3?
4. (Example of **Continuous infinite sample space**). If a point is randomly chosen in the plane inside the unit circle, what is the probability that it came from the inside the circle of radius $1/2$?

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- Sample space: How many ways can n people have their birthdays? Well, since there are 365 days, in total there are 365^n possibilities.
- How many ways are there for the n people to have distinct birthdays? The first person can be born any of the 365 days, then the second only has 364 choices, the third 363, and so on. So

  $$365 \cdot 364 \cdot \cdots \cdot (365 - n + 1).$$

  Therefore, we have that the probability of n people having distinct birthdays, P, is

  $$P = \frac{365 \cdot 364 \cdot \cdots \cdot (365 - n + 1)}{365^n},$$

  and thus

  $$p = 1 - P = 1 - \frac{365 \cdot 364 \cdot \cdots \cdot (365 - n + 1)}{365^n}.$$

One can compute that:

n	$=$	10	20	30	40	50	60
p	$=$	0.12	0.41	0.71	0.89	0.97	0.99