Christopher Croke
University of Pennsylvania

Lagrange Multipliers

Math 115

How to deal with constrained optimization.

How to deal with constrained optimization.

Let's revisit the problem of finding the cheapest box (with double bottom) of volume 324 cubic inches.

How to deal with constrained optimization.
Let's revisit the problem of finding the cheapest box (with double bottom) of volume 324 cubic inches. Minimize

$$f(x, y, z) = 3xy + 2xz + 2yz.$$

(**Objective function**)

How to deal with constrained optimization.
Let's revisit the problem of finding the cheapest box (with double bottom) of volume 324 cubic inches. Minimize

$$f(x, y, z) = 3xy + 2xz + 2yz.$$

(**Objective function**)
subject to

$$xyz = 324$$

(**constraint equation**).

How to deal with constrained optimization.

Let's revisit the problem of finding the cheapest box (with double bottom) of volume 324 cubic inches. Minimize

$$f(x, y, z) = 3xy + 2xz + 2yz.$$

(**Objective function**)
subject to

$$xyz = 324$$

(**constraint equation**).
Previously we used constraint equation to write z as a function of x and y. But you can't always do that.

Problem; Find the minimum value of $x^2 + y^2 + 1$ subject to $x + y - 2 = 0$. (use Maple to graph).

Problem; Find the minimum value of $x^2 + y^2 + 1$ subject to $x + y - 2 = 0$. (use Maple to graph).

General Problem: Find (x, y) that maximizes (or minimizes) $f(x, y)$ subject to $g(x, y) = 0$.

Problem; Find the minimum value of $x^2 + y^2 + 1$ subject to $x + y - 2 = 0$. (use Maple to graph).

General Problem: Find (x, y) that maximizes (or minimizes) $f(x, y)$ subject to $g(x, y) = 0$.

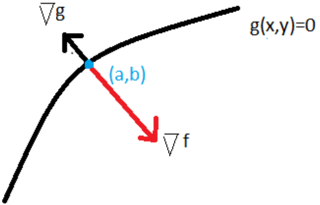
The answer: If (a, b) solves the problem then there is a number λ such that

$$\nabla f(a, b) = \lambda \nabla g(a, b) \quad \text{and} \quad g(a, b) = 0.$$

Problem; Find the minimum value of $x^2 + y^2 + 1$ subject to $x + y - 2 = 0$. (use Maple to graph).
General Problem: Find (x, y) that maximizes (or minimizes) $f(x, y)$ subject to $g(x, y) = 0$.
The answer: If (a, b) solves the problem then there is a number λ such that

$$\nabla f(a, b) = \lambda \nabla g(a, b) \quad \text{and} \quad g(a, b) = 0.$$

(Since ∇g is perpendicular to the level curves this says that ∇f is perpendicular to the level curve $g(x, y) = 0$ at the point (a, b).)



Method of Lagrange Multipliers:

Define

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y)$$

Method of Lagrange Multipliers:

Define

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y)$$

Now find all points that satisfy the three equations:

$$F_x(a, b, \lambda) = 0, \quad F_y(a, b, \lambda) = 0, \quad F_\lambda(a, b, \lambda) = 0.$$

Method of Lagrange Multipliers:

Define

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y)$$

Now find all points that satisfy the three equations:

$$F_x(a, b, \lambda) = 0, \qquad F_y(a, b, \lambda) = 0, \qquad F_\lambda(a, b, \lambda) = 0.$$

(Note the last equation is just $g(a, b) = 0$ which is our constraint equation.)

Method of Lagrange Multipliers:

Define

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y)$$

Now find all points that satisfy the three equations:

$$F_x(a, b, \lambda) = 0, \qquad F_y(a, b, \lambda) = 0, \qquad F_\lambda(a, b, \lambda) = 0.$$

(Note the last equation is just $g(a, b) = 0$ which is our constraint equation.)

Now let's do our example.

This method usually works:

- Solve both $F_x = 0$ and $F_y = 0$ for λ and equate the expressions.

This method usually works:

- Solve both $F_x = 0$ and $F_y = 0$ for λ and equate the expressions.
- Solve for one of x or y.

This method usually works:

- Solve both $F_x = 0$ and $F_y = 0$ for λ and equate the expressions.
- Solve for one of x or y.
- Plug into $F_\lambda = 0$ and solve for one variable.

This method usually works:

- Solve both $F_x = 0$ and $F_y = 0$ for λ and equate the expressions.
- Solve for one of x or y.
- Plug into $F_\lambda = 0$ and solve for one variable.
- Use result to find other two variables.

This method usually works:

- Solve both $F_x = 0$ and $F_y = 0$ for λ and equate the expressions.
- Solve for one of x or y.
- Plug into $F_\lambda = 0$ and solve for one variable.
- Use result to find other two variables.

Problem: For a firm with production function $f(x, y) = 20x^{\frac{2}{3}}y^{\frac{1}{3}}$, assume that a unit x of labor costs \$10 and a unit y of capital costs \$20. If the firm has \$12,000 to spend, how many units of labor and how many units of capital should it use to maximize production.

This method usually works:

- Solve both $F_x = 0$ and $F_y = 0$ for λ and equate the expressions.
- Solve for one of x or y.
- Plug into $F_\lambda = 0$ and solve for one variable.
- Use result to find other two variables.

Problem: For a firm with production function $f(x, y) = 20x^{\frac{2}{3}}y^{\frac{1}{3}}$, assume that a unit x of labor costs \$10 and a unit y of capital costs \$20. If the firm has \$12,000 to spend, how many units of labor and how many units of capital should it use to maximize production.
In this problem λ turns out to be the **marginal productivity of money**, i.e. One extra dollar should produce 0.83994 units of production.

Can do more variables. Lets try our box example
$f(x, y, z) = 3xy + 2xz + 2yz$ with $324 - xyz = 0$.

Can do more variables. Lets try our box example $f(x, y, z) = 3xy + 2xz + 2yz$ with $324 - xyz = 0$.

Can handle more constraints. (See book.)

Can do more variables. Lets try our box example $f(x, y, z) = 3xy + 2xz + 2yz$ with $324 - xyz = 0$.

Can handle more constraints. (See book.)

Can use to find max and min on a bounded region.

- Find critical points in the interior.

Can do more variables. Lets try our box example $f(x, y, z) = 3xy + 2xz + 2yz$ with $324 - xyz = 0$.

Can handle more constraints. (See book.)

Can use to find max and min on a bounded region.

- Find critical points in the interior.
- For each edge solve the constrained problem.

Can do more variables. Lets try our box example $f(x, y, z) = 3xy + 2xz + 2yz$ with $324 - xyz = 0$.

Can handle more constraints. (See book.)

Can use to find max and min on a bounded region.

- Find critical points in the interior.
- For each edge solve the constrained problem.
- Check vertices

Can do more variables. Lets try our box example $f(x, y, z) = 3xy + 2xz + 2yz$ with $324 - xyz = 0$.

Can handle more constraints. (See book.)

Can use to find max and min on a bounded region.

- Find critical points in the interior.
- For each edge solve the constrained problem.
- Check vertices
- Take the max (or min) of the first three.

Problem: Find the maximum and minimum values of $f(x, y) = y^2 - y + x^2 - 2$ on the upper half unit disc.