• Measuring the i

LECTURE 25
↳ PageRank /
Linear Prog

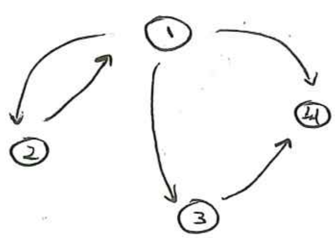
Many options: ex

ee" and 'out degree".

coming from

eting, networks, ...)

<u>PageRank</u>

G 

Iterative algorithm: Importance of each node is determined by the number of edges going in weighed by the importance of those nodes.

In particular, — the weights are: $\dfrac{1}{\text{n° of links going out}}$

$$\left.\begin{aligned}
p_1(k+1) &= p_2(k)\tfrac{1}{2}\\
p_2(k+1) &= p_1(k)\tfrac{1}{3} + p_2(k)\tfrac{1}{2}\\
p_3(k+1) &= p_1(k)\tfrac{1}{3}\\
p_4(k+1) &= p_1(k)\tfrac{1}{3} + p_3(k)
\end{aligned}\right\} \rightarrow$$

In matrix form:

$$\vec{p}(k+1) = \underbrace{\begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 0 & 1 & 0 \end{bmatrix}}_{\text{Transition matrix (A) (weighted adjacency matrix)}} \vec{p}(k)$$

a_{ij} = "probability" of going from node j to i

In general,

$$A \to a_{ij} = \begin{cases} \dfrac{1}{n_j} & \text{if there is an arrow from } j \text{ to } i, \text{ with } n_j = \text{n}^{\text{o}} \text{ of arrows going out of node } j. \\ 0 & \text{otherwise} \end{cases}$$

Problems: We cannot ensure that there is a solution or that it is unique.

↳ We would like to use the consequence of Perron-Frobenius for Markov positive matrix:

Th: A positive Markov matrix always has $\lambda = 1$ non-repeated, $|\lambda| < 1$ for all others, and therefore it has a unique steady state $u_\infty = A u_\infty$.

→ We need to fix two things: 1) "Dangling" nodes: those without outgoing arrows.

2) Zero entries.

• Solution to 1): Change the ~~col~~ column of zeros with the column $\begin{bmatrix} 1/n \\ \vdots \\ 1/n \end{bmatrix}$.

↳ You can think of this as follows: a random "surfer" will go to any website with some probability if there are no links in the current website.

In our example,

$$A_2 = \begin{bmatrix} 0 & 1/2 & 0 & 1/4 \\ 1/3 & 1/2 & 0 & 1/4 \\ 1/3 & 0 & 0 & 1/4 \\ 1/3 & 0 & 1 & 1/4 \end{bmatrix}$$

Remark: By construction, after this step we always will have a Markov matrix → cols. add up to 1.

• Solution to 2): "Damping" the matrix A:

$$A_\alpha = (1-\alpha) A_2 + \frac{\alpha}{n} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}, \quad \alpha \in [0,1]$$

Meaning: From any website you can go to any other with some small probability.

↳ If $\alpha \approx 1$, we lose the structure of A.

MATH 312
LECTURE 23

: Linear Programming

Example: You have 1\$, and you have to invest it on assets A and B, satisfying:

1) You invest in A at least ~~twice~~ double than in B.

2) At least 1/2 \$ is invested.

We can write this constraints as inequalities:

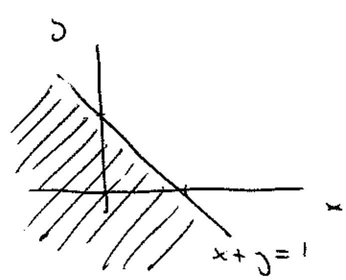
Denote x = invested on A, y = " on B, then

$$\left.\begin{array}{ll} x+y \le 1 & \text{(you only have 1\$)} \\ x \ge 2y & \text{(at least double in A than B)} \\ x+y \ge \frac{1}{2} & \text{(at least ½\$ invested)} \end{array}\right\} \text{ and } \left\{\begin{array}{l} x \ge 0 \\ y \ge 0 \end{array}\right.$$

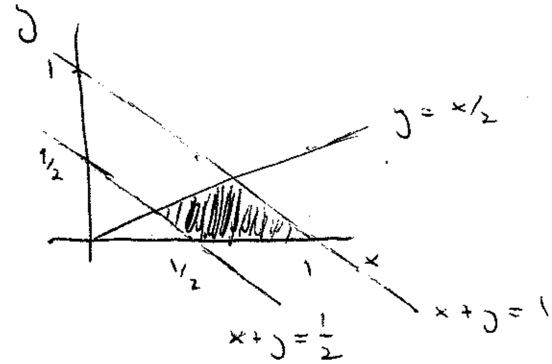
Q: What values are x, y allowed to take?

Note that each inequality represent a half plane:

$x+y \le 1 \rightarrow$



Drawing all the constraints gives:



This is the <u>feasible</u> region.

We usually denote all the constraints in matrix form:

$A\vec{x} \le \vec{b}$.

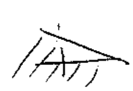
Q: What for $x + y \ge \frac{1}{2}$? $\rightarrow$ $-x - y \le -\frac{1}{2}$.

So, $\left.\begin{array}{l} x + y \le 1 \\ -x + 2y \le 0 \\ -x - y \le -\frac{1}{2} \end{array}\right\} \rightarrow \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \le \begin{bmatrix} 1 \\ 0 \\ -1/2 \end{bmatrix}$ (and $x \ge 0, y \ge 0$).

Def: Feasible region

Intersection of the half-spaces arising from the inequalities.

<u>Remark</u>: It can be · bounded : $x + y \le 1, x \ge 0, y \ge 0$ 

· Unbounded: $x + y \le 1$ 

· empty $\emptyset$ $x + y \ge 1, x + y \le 0$.

Def. A linear Programming problem asks to maximize a linear profit $\vec{c}^T \vec{x}$ subject to linear constraints $A\vec{x} \le \vec{b}$ and $\vec{x} \ge 0$, where $A, \vec{c}, \vec{b}$ are data.

- In our example: Say that A has a twofold return while threefold in B.

  Then, for an investment of x in A, y in B, we get back

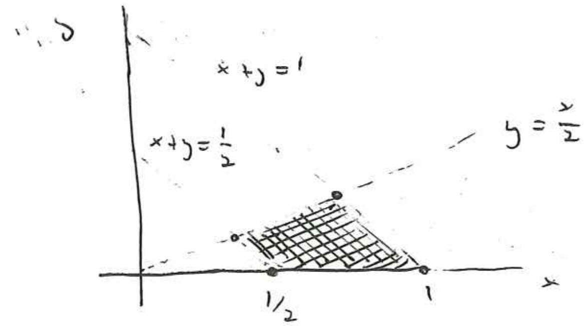
  $$2x + 3y = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \to \vec{c}^T = \begin{bmatrix} 2 & 3 \end{bmatrix}$$

- So the lin. programming problem would be:

$$\underbrace{\left.\begin{array}{ll} \max. & \vec{c}^T \vec{x} \\ \text{s.t.} & A\vec{x} \le \vec{b} \\ & \vec{x} \ge 0 \end{array}\right\}}_{\text{Standard form}} \text{ where } \quad \vec{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \; A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ -1 & -1 \end{bmatrix}, \; \vec{b} = \begin{bmatrix} 1 \\ 0 \\ 1/2 \end{bmatrix}, \; \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} \text{ unknowns.}$$

- How to solve it? → Geometric idea (we'll follow our example).

1) Draw the feasible region:

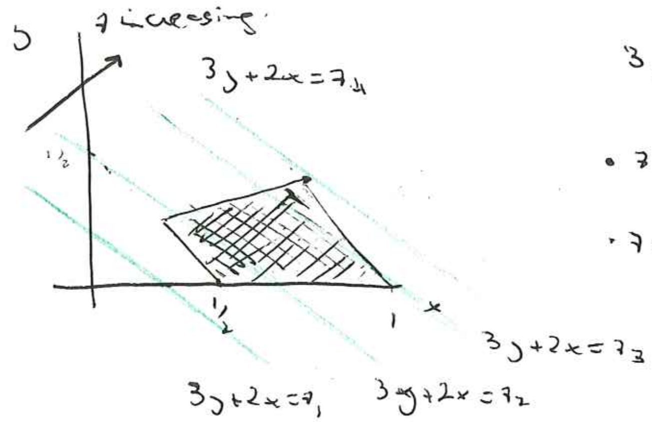


All possible solution are in that region.

2) Maximize the profit: $\vec{c}^T\vec{x} = [2 \ 3]\begin{bmatrix} x \\ y \end{bmatrix} = 2x+3y$

↳ Pick a profit z ⇝ then $z = 2x+3y$ describes a line;

each combination of x, y on that line gives same profit.



$3y+2x=z \to y = \frac{z}{3} - \frac{2}{3}x$

- $z=0 \rightsquigarrow y = \frac{-2}{3}x$
- $z=1 \rightsquigarrow y = \frac{1}{3} - \frac{2}{3}x$

⋮

We "move" the line that gives the profit until we leave the feasible region. We can see that the maximum will always be obtained at a corner (*).

(*) Assuming the feasible region is bounded and that there is a unique solution.

3) Evaluate the profit function $\vec{c}^T \vec{x}$ at all corners and pick the maximum.

In our example,

$(x=\frac{1}{2}, y=0)$, $\quad \begin{matrix} x+y=\frac{1}{2} \\ 2y-x=0 \end{matrix} \Big\} \Rightarrow (y=\frac{1}{6}, x=\frac{1}{3})$

$(x=1, y=0)$, $\quad \begin{matrix} x+y=1 \\ 2y-x=0 \end{matrix} \Big\} \rightarrow (y=\frac{1}{3}, x=\frac{2}{3})$

$\hookrightarrow (\frac{1}{2}, 0) \Rightarrow 2\frac{1}{2} + 3\cdot 0 = 1$

$(1,0) \rightarrow 2$

$(\frac{1}{3}, \frac{1}{6}) \rightarrow 2\cdot\frac{1}{3} + 3\frac{1}{6} = \frac{7}{6}$

$(\frac{2}{3}, \frac{1}{3}) \Rightarrow 2\cdot\frac{2}{3} + 3\frac{1}{3} = \frac{4}{3} + 1 = \frac{7}{3}$

$\Rightarrow$ maximum is $\frac{7}{3}$, at point $x=\frac{2}{3}$, $y=\frac{1}{3}$.

(We "saw" it on the picture).

<u>Remark</u>: In higher dimensions, finding all the corners and evaluating the profit is <u>not</u> an efficient method. But the moral is true: the optimum is reached at a corner.

Problem 6 [10 points]

Part a. A small brewery produces ale and lager beers. The production is limited by scarce resources: they only have available 450 lbs of corn and 1200 lbs of barley malt. Nevertheless, they have to produce at least 10 barrels of beers (either ale or lager). Producing 1 barrel of ale requires 5 lbs of corn and 40 lbs of barley malt. On the other hand, producing 1 barrel of lager requires 15 lbs of corn and 20 lbs of barley malt. They obtain a profit of \$10 for each barrel of ale and \$20 for each barrel of lager. Their goal is to maximize the profit.

Write this as a linear programming problem in standard form. Graph the feasible domain and find the solution to the LP using geometric methods.

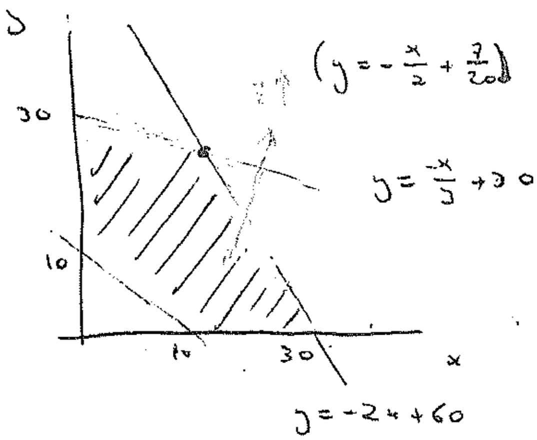
P6

a) Standard form: ($x \equiv$ nº barrels of ale, $y \equiv$ nº " of lager)

$$\left.\begin{aligned} \max\ & 10x + 20y \\ \text{st.}\ & -x-y \le -10 \\ & 5x + 15y \le 450 \\ & 40x + 20y \le 1200 \\ & x, y \ge 0 \end{aligned}\right\} \quad (\leftarrow x+y \ge 10)$$

Graph: $x+y=10$, $y = \frac{-5}{15}x + \frac{450}{15} = \frac{-x}{3} + 30$

$y = -2x + 60$



Candidates: $(0,10)$, $(0,30)$, $(10,0)$, $(30,0)$

and intersection of

$$\left.\begin{aligned} y &= \frac{-x}{3} + 30 \\ y &= -2x + 60 \end{aligned}\right\} \to 0 = -2x + 60 + \frac{x}{3} - 30 \to$$

$\to \frac{5}{3}x = 30 \Rightarrow \underline{x = 18}$

$y = -6 + 30 = 24$

$(18, 24)$.

Optimum: $z = 10 \cdot 18 + 20 \cdot 24 = 180 + 480 = \boxed{660 = z}$

$\boxed{\text{at } x = 18,\ y = 24}$

(Others:

$z = 100$

$z = 300$

$z = 200$

$z = 600$)