MATH 425
LECTURE 21

: Harmonic functions / Fourier transform.

(Last day)

Solution to Dirichlet problem for a circle,

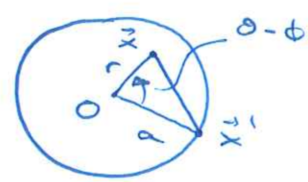
$$\left.\begin{array}{ll}\Delta u = 0, & r < a \\ u(a,\theta) = h(\theta) & \end{array}\right\}$$ is given by the Poisson's formula

$$u(r,\theta) = \frac{a^2 - r^2}{2\pi}\int_0^{2\pi} \frac{h(\phi)}{a^2 + r^2 - 2ar\cos(\theta-\phi)}\,d\phi.$$

- Notice that we can write this back in (x,y):

$$|\vec{x} - \vec{x}'|^2 = a^2 + r^2 - 2ar\cos(\theta-\phi)$$

(cosines law)



$\downarrow$

$$\boxed{u(\vec{x}) = \frac{a^2 - |\vec{x}|^2}{2\pi}\int_{|\vec{x}'|=a} \frac{u(\vec{x}')}{|\vec{x} - \vec{x}'|^2}\,\frac{ds}{a}}$$

- We will now use Poisson's formula to prove the strong maximum principle and that harmonic functions are smooth (away from the boundary).

- Theorem (Poisson's formula)

  Let $h(\theta)$ be a continuous function ($\theta \in [0, 2\pi)$). Then, if we denote $w(\vec{x}') = h(\theta)$, we have that

1) $$u(\vec{x}) = \frac{a^2 - |\vec{x}|^2}{2\pi} \int_{|\vec{x}'| = a} \frac{w(\vec{x}')}{|\vec{x} - \vec{x}'|^2} \frac{ds}{a}$$ is harmonic in D (circle of radious a). (i.e., $\Delta u = 0$ in D).

2) It is the only harmonic function in D that satisfies

$$\lim_{\vec{x} \to \vec{x}_0} u(\vec{x}) = w(\vec{x}_0) \equiv h(\theta_0) \text{ for all } \vec{x}_0 \in C \quad (C \equiv \partial D).$$

That is, u is harmonic in D and continuous in $\bar{D} = D \cup \partial D$, satisfying the BC.

(proof in book: not required)

• Proposition: (Mean value property)

Let u be harmonic in a disk D, continuous in $\bar{D} = D \cup \partial D$ (disk of radius a) } Then,

$$\left\| \begin{matrix} \text{the value of } u \\ \text{at the center of } D \end{matrix} = \begin{matrix} \text{average of } u \\ \text{on } \partial D \end{matrix} \right\|.$$

Proof: (Since Δ is invariant under traslations, given the center $\vec{x}_c$ change coordinates to make it the origin).

Using Poisson's formula at $\vec{x} = \vec{0}$,

$$u(\vec{0}) = \frac{\cancel{a^2}}{2\pi} \int_{|\vec{x}'|=a} \frac{u(\vec{x}')}{\cancel{|\vec{x}'|^2}} \frac{ds}{a} = \frac{1}{2\pi a} \int_{|\vec{x}'|=a} u(\vec{x}')\, ds \; /\!/ .$$

• We are now ready for the strong max principle.

Proof: $\left[\begin{matrix} \Delta u = 0 \text{ in } D \text{ (connected, bounded, open)} \\ u \text{ continuous on } \bar{D} \end{matrix} \right\} \Rightarrow$ max/min attained on ∂D and nowhere inside $\Big]$

We already proved the weak form, i.e.,

that the maximum was attained at some point $\vec{x}_* \in \partial D$.

Say now the maximum is also attained at $\vec{x}_M \in \bar{D}$. We have to show that $\vec{x}_M \notin D$ (unless $u \equiv$ constant).

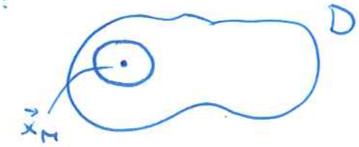
We would have that

$$u(\vec{x}) \leq u(\vec{x}_M) = M \quad \forall \vec{x} \in D.$$

$\uparrow$ max value

if $\vec{x}_M \in D$

But then, we can draw a circle around $\vec{x}_M$, completely inside D (it was an open set), so $u(\vec{x}_M)$ has to be equal to its average around that circle:



By definition of $\vec{x}_M$, at no point on the circumf. u can be bigger than M. So, for the average to be equal to $u(\vec{x}_M) = M$, the only choice left is that $u(\vec{x}) = M$ for all $\vec{x}$ on that circumference. (* we are using implicitly that u is continuous).

But this is true for any circle centered at $\vec{x}_M$ and contained in D (different radious). Moreover, now we can pick a different point $\vec{x}_{M_2}$ on one of those circles, and repeat the argument, filling D in that way (D connected).

• <u>Proposition:</u> (Smoothness of harmonic functions).

Let u be harmonic in an open set $\Omega \subset \mathbb{R}^2$. Then, $u \in C^\infty$ in D, i.e., all the partial derivatives of all orders exists (and are continuous).

Proof,

Consider first $\Omega = D$ (disk centered at origin, radius a).
Then,

$$u(\vec{x}) = \frac{a^2 - |\vec{x}|^2}{2\pi} \int_{|\vec{x}'|=a} \frac{u(\vec{x}')}{|\vec{x} - \vec{x}'|^2} \frac{ds}{a}.$$

Derivatives in $\vec{x}$ only affect the denominator of the integrand (when differentiating under the integral sign), which is differentiable to all orders outside ∂D (notice $|\vec{x} - \vec{x}'| \neq 0$ if $\vec{x} \in D$, since $\vec{x}' \in \partial D$).

Finally, for a general domain Ω and $\vec{x}_0 \in \Omega$, we simply consider a circle around $\vec{x}_0$ (and contained in Ω) and repeat the argument. // (Ω needs to be open).

Fourier transforms

Motivation and definition

Up to now, we've been expressing functions on finite intervals (usually the interval $0 \leqslant x \leqslant L$ or $-L \leqslant x \leqslant L$) as Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

and

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

We also occasionally thought about the complex exponential version of Fourier series: Since $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$, or equivalently

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i},$$

we can rewrite the above series as:

$$\begin{aligned} f(x) &= a_0 e^{0ix} + \sum_{n=1}^{\infty} a_n \frac{e^{n\pi ix/L}+e^{-n\pi ix/L}}{2} + b_n \frac{e^{n\pi ix/L}-e^{-n\pi ix/L}}{2i} \\ &= a_0 e^{0ix} + \sum_{n=1}^{\infty} \frac{a_n+ib_n}{2} e^{-n\pi ix/L} + \frac{a_n-ib_n}{2i} e^{n\pi ix/L} \\ &= \sum_{n=-\infty}^{\infty} c_n e^{n\pi ix/L} \end{aligned}$$

where

$$c_n = \begin{cases} \frac{1}{2}(a_n - ib_n) & \text{for } n > 0 \\ a_0 & \text{for } n = 0 \\ \frac{1}{2}(a_{-n} + ib_{-n}) & \text{for } n < 0 \end{cases}$$

Using the formulas for a_n and b_n given above, we see that, for $n > 0$.

$$\begin{aligned}
c_n &= \frac{1}{2}(a_n - ib_n) \\
&= \frac{1}{2L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx - \frac{i}{2L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \\
&= \frac{1}{2L}\int_{-L}^{L} f(x)\left[\cos\left(\frac{n\pi x}{L}\right) - i\sin\left(\frac{n\pi x}{L}\right)\right]\,dx \\
&= \frac{1}{2L}\int_{-L}^{L} f(x)e^{-n\pi ix/L}\,dx.
\end{aligned}$$

If $n < 0$ we have

$$\begin{aligned}
c_n &= \frac{1}{2}(a_{-n} + ib_{-n}) \\
&= \frac{1}{2L}\int_{-L}^{L} f(x)\cos\left(-\frac{n\pi x}{L}\right)\,dx + \frac{i}{2L}\int_{-L}^{L} f(x)\sin\left(-\frac{n\pi x}{L}\right)\,dx \\
&= \frac{1}{2L}\int_{-L}^{L} f(x)\left[\cos\left(\frac{n\pi x}{L}\right) - i\sin\left(\frac{n\pi x}{L}\right)\right]\,dx \\
&= \frac{1}{2L}\int_{-L}^{L} f(x)e^{-n\pi ix/L}\,dx
\end{aligned}$$

because cosine is an even function and sine is odd. So the same formula works for all the coefficients (even c_0) in this case and we have

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{n\pi ix/L} \quad \text{where} \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-n\pi ix/L}\,dx.$$

Equivalently, we could write:

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2L} c_n e^{n\pi ix/L} \quad \text{where} \quad c_n = \int_{-L}^{L} f(x)e^{-n\pi ix/L}\,dx.$$

What we want to do here is let L tend to infinity, so we can consider problems on the whole real line. To see what happens to our Fourier series formulas when we do this, we introduce two new variables: $\omega = n\pi/L$ and $\Delta\omega = \pi/L$. Then our complex Fourier series formulas become

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} c_n e^{i\omega x} \quad \text{where} \quad c_n = \int_{-L}^{L} f(x)e^{-i\omega x}\,dx$$

and the n in the formula for c_n is hiding in the variable ω. We can rewrite these as

$$f(x) = \sum_{n=-\infty}^{\infty} c_\omega e^{i\omega x}\frac{\Delta\omega}{2\pi} \quad \text{where} \quad c_\omega = \int_{-L}^{L} f(x)e^{-i\omega x}\,dx.$$

The variable $\omega = n\pi/L$ takes on more and more values which are closer and closer together as $L \to \infty$, so c_ω begins to feel like a function of the variable ω defined for all real ω. Likewise, the sum on the left looks an awful lot like a Riemann sum approximating an integral. What happens in the limit as $L \to \infty$ is:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c(\omega)e^{i\omega x}\,d\omega \quad \text{where} \quad c(\omega) = \int_{\infty}^{\infty} f(x)e^{-ix\omega}\,dx.$$

The formula on the right defines the function $c(\omega)$ as the *Fourier transform* of $f(x)$, and the formula on the left defines $f(x)$ as the *inverse Fourier transform* of $c(\omega)$.

Fourier transform: $\widehat{f}(\omega) = F(\omega) = \mathcal{F}[f(x)](\omega) = \int_{-\infty}^{\infty} f(x)e^{-ix\omega}\,dx$

Inverse Fourier transform: $\check{F}(x) = f(x) = \mathcal{F}^{-1}[F(\omega)](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{ix\omega}\,d\omega$

These formulas hold true (and the inverse Fourier transform of the Fourier transform of $f(x)$ is $f(x)$ — the so-called *Fourier inversion formula*) for reasonable functions $f(x)$ that decay to zero as $|x| \to \infty$ in such a way so that $|f(x)|$ and/or $|f(x)|^2$ has a finite integral over the whole real line.

There are many standard notations for Fourier transforms (and alternative definitions with the minus sign in the Fourier transform rather than in the inverse, and with the 2π factor in different places, so watch out if you're looking in books other than our textbook!), including

$$\widehat{f}(\omega) = F(\omega) = \mathcal{F}[f(x)](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{ix\omega}\,dx$$

and

$$\check{F}(x) = f(x) = \mathcal{F}^{-1}[F(\omega)](x) = \int_{-\infty}^{\infty} F(\omega)e^{-ix\omega}\,d\omega$$

(which is the one in the current Math 241 textbook, I think).

Properties and examples.

The Fourier transform is an operation that maps a function of x, say $f(x)$ to a function of ω, namely $\mathcal{F}[f](\omega) = \hat{f}(\omega)$. It is clearly a *linear* operator, so for functions $f(x)$ and $g(x)$ and constants α and β we have

$$\mathcal{F}\left[\alpha f(x) + \beta g(x)\right] = \alpha\mathcal{F}\left[f(x)\right] + \beta\mathcal{F}\left[g(x)\right].$$

Some other properties of the Fourier transform are

1. **Translation** (or shifting): $\mathcal{F}\left[f(x-a)\right](\omega) = e^{-i\omega a}\mathcal{F}\left[f(x)\right](\omega)$. And in the other direction, $\mathcal{F}\left[e^{iax}f(x)\right](\omega) = \mathcal{F}\left[f(x)\right](\omega - a)$.
2. **Scaling**: $\mathcal{F}\left[\frac{1}{a}f\left(\frac{x}{a}\right)\right](\omega) = \mathcal{F}\left[f(x)\right](a\omega)$, and likewise $\mathcal{F}\left[f(ax)\right](\omega) = \frac{1}{a}\mathcal{F}\left[f(x)\right]\left(\frac{\omega}{a}\right)$.
3. **Operational property** (derivatives): $\mathcal{F}\left[f'(x)\right](\omega) = i\omega\mathcal{F}\left[f(x)\right](\omega)$, and $\mathcal{F}\left[xf(x)\right](\omega) = i\frac{d}{d\omega}\left(\mathcal{F}\left[f(x)\right](\omega)\right)$.

The operational property is of essential importance for the study of differential equations, since it shows that the Fourier transform converts derivatives to multiplication – so it converts calculus to algebra (or might reduce a partial differential equation to an ordinary one).

Here are the proofs of the first of each of the three pairs of formulas to give a sense of how to work with Fourier transforms, and leave the other three as exercises. For the first shifting rule, we make the substitution $y = x - a$ (so $dy = dx$ and $x = y + a$) to calculate

$$\begin{aligned}\mathcal{F}\left[f(x-a)\right](\omega) &= \int_{-\infty}^{\infty} f(x-a)e^{-i\omega x}\,dx \\ &= \int_{-\infty}^{\infty} f(y)e^{-i\omega y}e^{-i\omega a}\,dx \\ &= e^{-i\omega a}\mathcal{F}\left[f(x)\right](\omega)\end{aligned}$$

For the first scaling rule, we make the substitution $y = x/a$ (so $dx = a\,dy$) and get

$$\begin{aligned}\mathcal{F}\left[\frac{1}{a}f\left(\frac{x}{a}\right)\right](\omega) &= \int_{-\infty}^{\infty} \frac{1}{a}f\left(\frac{x}{a}\right)e^{-i\omega x}\,dx \\ &= \int_{-\infty}^{\infty} f(y)e^{-ia\omega y}\,dy \\ &= \mathcal{F}\left[f(x)\right](a\omega)\end{aligned}$$

For the operational property we first point out that since the Fourier transforms of both $f'(x)$ and $f(x)$ exist, we must have that $f(x) \to 0$ and $f'(x) \to 0$ as $x \to \pm\infty$. Therefore the endpoint terms will vanish when we integrate by parts (with $u = e^{-i\omega x}$ and $dv = f'(x)\,dx$, so $du = -i\omega e^{-i\omega x}$ and $v = f(x)$):

$$\begin{aligned}\mathcal{F}\left[f'(x)\right](\omega) &= \int_{-\infty}^{\infty} f'(x)e^{-i\omega x}\,dx \\ &= e^{-i\omega x}f(x)\Big|_{x=-\infty}^{x=\infty} + \int_{-\infty}^{\infty} i\omega f(x)e^{-i\omega x}\,dx \\ &= 0 + i\omega\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx \\ &= i\omega\mathcal{F}\left[f(x)\right](\omega)\end{aligned}$$