MATH 312
LECTURE 19

: The spectral theorem.

[Continuation ODEs and diff. eq.]

- <u>Example:</u> ODE with complex eigenvalues.

Solve $\vec{x}'(t) = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \vec{x}(t)$, for general initial data $\vec{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

<u>Sol</u>:

$\vec{x}(t) = e^{At}\vec{x}_0.$

[* <u>Remark</u>: You might prefer to use the method described in the 3y book]

$\det(A - \lambda I) = 0 \rightarrow \lambda_1 = 2+3i$

$\lambda_2 = 2-3i.$

- $\lambda_1 = 2+3i \rightarrow \vec{u}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$, $\cdot\ \lambda_2 = 2-3i \rightarrow \vec{u}_2 = \begin{bmatrix} i \\ -1 \end{bmatrix}$

- $e^{At} = \begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{2t}e^{3it} & 0 \\ 0 & e^{2t}e^{-3it} \end{bmatrix} \begin{bmatrix} -i & 1 \\ -i & -1 \end{bmatrix} \frac{1}{2} =$

$$= \frac{1}{2} \begin{bmatrix} e^{2t}e^{3ti} + e^{2t}e^{-3it} & ie^{2t}e^{3ti} - ie^{2t}e^{-3ti} \\ -ie^{2t}e^{3it} + ie^{2t}e^{-3it} & e^{2t}e^{3it} + e^{2t}e^{-3it} \end{bmatrix} \Rightarrow \vec{x}(t) = e^{At}\vec{x}_0 \Rightarrow$$

$$\Rightarrow \vec{x}(t) = \begin{bmatrix} \frac{1}{2}x_0 e^{2t}\left(\cos(3t) + i\cancel{\sin(3t)} + \cos(3t) - i\cancel{\sin(3t)}\right) + \frac{1}{2}y_0 e^{2t}\left(i\cancel{\cos(3t)} - i\sin(3t) + -i\cancel{\cos(3t)} - \sin(3t)\right) \\ \frac{1}{2}x_0 e^{2t}\left(-i\cancel{\cos(3t)} + \sin(3t) + i\cancel{\cos(3t)} + \sin(3t)\right) + \frac{1}{2}y_0 e^{2t}\left(\cos(3t) + i\cancel{\sin(3t)} + \cos(3t) + -i\cancel{\sin(3t)}\right) \\ \vdots \end{bmatrix}$$

VI/ The Spectral Theorem.

• Theorem: The Spectral Theorem
(or Principal Axis Theorem)

If M is a real symmetric $n\times n$ matrix, then it has real eigenvalues and the eigenvectors can be chosen to be orthonormal. That is,

M can always diagonalised as

$M = Q D Q^T$, where

$D \equiv$ diagonal with real eigenvalues.

$Q \equiv$ formed by n orthonormal eigenvectors.

Remarks:

1) Eigenvectors corresponding to eigenvalues go in the same order:

$$\begin{matrix}\lambda_1 \leftrightarrow \vec{u}_1 \\ \lambda_2 \leftrightarrow \vec{u}_2 \\ \vdots\end{matrix} \rightsquigarrow Q = \left[\vec{u}_1 | \vec{u}_2 | \dots\right], \quad D = \begin{bmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \ddots\end{bmatrix}$$

Note that if $M = QDQ^T$, then M is symmetric: $(QDQ^T)^T = QDQ^T$.

So $M = QDQ^T \Leftrightarrow M$ is symmetric

2) If the eigenvalues are distinct, then the eigenvectors are automatically perpendicular to each other.

↳ We just need to be careful to make them unit size.
(i.e., orthonormal).

3) If an eigenvalue is repeated (let's say twice), the the nullspace of $A-\lambda I$ will have dimension 2 (or more if repeated more times...).

In this case we need to obtain an orthonormal basis of that nullspace (the "eigenspace").

↳ obtain a basis and use Gram-Schmidt!
(as usual)

4) Q has orthonormal columns and it is square. Thus, it automatically has ~~orthogonal~~ orthonormal rows too. This is called orthogonal matrix. [(*) proof in -163-]

Q orthogonal $\Leftrightarrow Q^TQ = QQ^T = I$ (i.e., $Q^T = Q^{-1}$).

Q orthonormal columns $\Leftrightarrow Q^TQ = I$

Q orthonormal rows $\Leftrightarrow QQ^T = I$.

5) $M = QDQ^T = \left[\vec{q}_1 | \cdots | \vec{q}_n\right] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{q}_1^{\,T} \\ \vdots \\ \vec{q}_n^{\,T} \end{bmatrix} =$

$$= \left[\lambda_1 \vec{q}_1 | \cdots | \lambda_n \vec{q}_n\right] \begin{bmatrix} \vec{q}_1 \\ \vdots \\ \vec{q}_n \end{bmatrix} =$$

$$= \boxed{\lambda_1 \underbrace{\vec{q}_1 \vec{q}_1^{\,T}} + \cdots + \lambda_n \underbrace{\vec{q}_n \vec{q}_n^{\,T}} = M} \quad \left(\begin{array}{c}\text{nice} \\ \text{decomposition}\end{array}\right)$$

projection matrices onto the (unitary) axis given by the (orthonormal) eigenvectors.

(*) Proof of: Q square with orthonormal columns $\to$ Q has orthonormal rows (and thus, Q is "orthogonal")

$\hookrightarrow$ $Q^TQ = I$. Then, we know that Q^{-1} exists since

$\det(Q^TQ) = \det(Q^T)\det(Q) = (\det(Q))^2 = \det(I) = 1 \Rightarrow$

$\Rightarrow \det(Q) = \pm 1 \neq 0$. (We needed determinants to prove this!)

Therefore, multiplying by Q we obtain

$$QQ^TQ = Q \Rightarrow QQ^TQQ^{-1} = QQ^{-1} \Rightarrow \underbrace{QQ^T = I}_{\Updownarrow}$$

orthonormal rows.

Before proving the theorem, we need to ~~remind~~ remember some things about complex numbers:

$z \in \mathbb{C} \rightarrow z = a + bi,\ a, b \in \mathbb{R},\ i^2 = -1$.

- $z_1 + z_2 = (a+bi) + (c+di) \underset{\substack{\uparrow \\ \text{def.}}}{=} (a+c) + (b+d)i$.
- $z_1 z_2 = (a+bi)(c+di) \underset{\substack{\uparrow \\ \text{def.}}}{=} (ac-bd) + i(ad+bc)$.
- $\bar{z}_1 \underset{\substack{\uparrow \\ \text{def.}}}{=} a - bi$ (conjugation)

Some more things we need:

1) $z + \bar{z} = a+bi + a-bi = 2a \in \mathbb{R}$ purely real.

2) $z - \bar{z} = 2ib$ purely imaginary

3) $z = \bar{z} \iff z \in \mathbb{R}$.

4) $z\bar{z} = a^2 + b^2 = |z|^2$ (modul of z)

5) $\vec{u} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \longrightarrow \|\vec{u}\|^2 = \bar{\vec{u}}^T \vec{u} = [\bar{z}_1\ \ \bar{z}_2] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = |z_1|^2 + |z_2|^2$.

6) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

<u>Proof of Spectral theorem</u>: Let's split it into two parts.

1) M real and symmetric $\Rightarrow$ real eigenvalues.

Let $\lambda, \vec{u}$ be an eigenvalue and its eigenvector (both maybe complex)

Then (We want to prove λ is real, i.e. $\lambda = \bar{\lambda}$)

$\boxed{1}$ $M\vec{u} = \lambda\vec{u}$

Therefore, taking conjugates, $\overline{M\vec{u}} = \overline{\lambda\vec{u}} \Rightarrow \bar{M}\bar{\vec{u}} = \bar{\lambda}\bar{\vec{u}} \Rightarrow M\bar{\vec{u}} = \bar{\lambda}\bar{\vec{u}}$

M is real.

$\boxed{2}$ $M\bar{\vec{u}} = \bar{\lambda}\bar{\vec{u}}$

$\rightarrow$ Multiply $\boxed{1}$ by $\bar{\vec{u}}^T$: $\bar{\vec{u}}^T M\vec{u} = \bar{\vec{u}}^T\lambda\vec{u} = \lambda\bar{\vec{u}}^T\vec{u} = \lambda\|\vec{u}\|^2$

$\rightarrow$ Multiply $\boxed{2}$ by $\vec{u}^T$: $\vec{u}^T M\bar{\vec{u}} = \vec{u}^T\bar{\lambda}\bar{\vec{u}} = \bar{\lambda}\vec{u}^T\bar{\vec{u}} = \bar{\lambda}\|\vec{u}\|^2$ $(*)$

These are just numbers! (not vectors or matrices)

[   ][   ][ ] = [   ][ ] = number

Therefore,

$$\underbrace{\bar{\vec{u}}^T M\vec{u} = (\bar{\vec{u}}^T M\vec{u})^T}_{\text{same number!}} = \vec{u}^T M^T\bar{\vec{u}} \underset{M \text{ is symmetric}}{=} \vec{u}^T M\bar{\vec{u}}$$

Thus we have found that

$\bar{\vec{u}}^T M \vec{u} = \vec{u}^T M \bar{\vec{u}}$ and we have in (*) that

$$\left.\begin{aligned} \bar{\vec{u}}^T M \vec{u} &= \lambda \|\vec{u}\|^2 \\ \vec{u}^T M \bar{\vec{u}} &= \bar{\lambda} \|\vec{u}\|^2 \end{aligned}\right\}$$

so we conclude that

$$\lambda \|\vec{u}\|^2 = \bar{\lambda} \|\vec{u}\|^2 \Rightarrow (\lambda - \bar{\lambda}) \|\vec{u}\|^2 \Rightarrow \lambda - \bar{\lambda} = 0 \Rightarrow \underline{\underline{\lambda = \bar{\lambda}}}$$

($\vec{u} \neq \vec{0}$ by definition of eigenvector)

($\Leftrightarrow \lambda$ is real)

2) M real, symmetric $\Rightarrow$ orthogonal eigenvectors.

Γ We only prove it for distinct eigenvalues.

But it is always true: if we have an eigenvalue repeated, for symmetric matrices we always obtain a full set of eigenvectors $\underset{\text{Gram-Schmidt}}{\overset{\text{so}}{\rightsquigarrow}}$ orthogonal set $\lrcorner$

Let $\lambda_1 \neq \lambda_2$ eigenvalues with corresponding eigenvectors $\vec{u}_1, \vec{u}_2$.

[Remark: We know from 1) that λ_1 and λ_2 are real (and thus $\vec{u}_1, \vec{u}_2$, being solutions of $(A-\lambda_1 I)\vec{u}_1 = \vec{0}$ ..., are also real).]

→ We want to show that $\vec{u}_1 \cdot \vec{u}_2 = \vec{u}_1^T \vec{u}_2 = \vec{u}_2^T \vec{u}_1 = 0$:

$$\left.\begin{array}{l} M\vec{u}_1 = \lambda_1 \vec{u}_1 \rightarrow \vec{u}_2^T M \vec{u}_1 = \lambda_1 \vec{u}_2^T \vec{u}_1 \\ M\vec{u}_2 = \lambda_2 \vec{u}_2 \rightarrow \vec{u}_1^T M \vec{u}_2 = \lambda_2 \vec{u}_1^T \vec{u}_2 \\ \text{but } \underset{\substack{\uparrow \\ \text{same number}}}{\vec{u}_2^T M \vec{u}_1} = (\vec{u}_2^T M \vec{u}_1)^T = \vec{u}_1^T M^T \vec{u}_2 \underset{\substack{\uparrow \\ M \text{ symmetric}}}{=} \vec{u}_1^T M \vec{u}_2 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \lambda_1 \vec{u}_2^T \vec{u}_1 = \lambda_2 \vec{u}_1^T \vec{u}_2 \Rightarrow (\lambda_1 - \lambda_2)\vec{u}_2^T \vec{u}_1 = 0 \underset{\substack{\uparrow \\ \lambda_1 \neq \lambda_2}}{\Rightarrow} \vec{u}_2^T \vec{u}_1 = 0 \,/\!/ .$$