Math 425/AMCS 525
Problems from previous exams

Problem 1: Solve $u_x + yu_y + u = 0$, $u(0, y) = y$. In what domain in the plane is your solution valid?

Problem 2 : Let $u(x, t)$ be the temperature in a rod of lenght L that satisfies the partial differential equation:

$$u_t = ku_{xx}, \quad (x, t) \in (0, L) \times (0, \infty),$$

where k is a positive constant, together with the initial condition

$$u(x, 0) = \phi(x), \quad x \in [0, L],$$

where ϕ satisfies $\phi(0) = \phi(L) = 0$ and $\phi(x) > 0$ for $x \in (0, L)$.

a) If u also satisfies the Neumann boundary conditions

$$u_x(0, t) = 0, \quad u_x(L, t) = 0,$$

show that the average temperature in the rod at time t, which is given by

$$A(t) = \frac{1}{L} \int_0^L u(x, t) dx,$$

is a constant (independent of time).

b) On the other hand, if u satisfies the Dirichlet boundary conditions

$$u(0, t) = 0, \quad u(L, t) = 0,$$

show that it must be the case that $u(x, t) \geq 0$ for all (x, t) satisfying $0 < x < L$ and $t > 0$.

c) Still under the assumption that u satisfies the Dirichlet boundary conditions, show that $A(t)$ is a non-increasing function of t.

Hint for a) and c): Use an argument similar to an energy argument.

Problem 3: Find as general a solution $u(x, y, z)$ as you can to the third-order equation

$$u_{xyz} = 0.$$

Problem 4 Solve the following initial-value problem for $u(x, y)$:

$$yu_x + u_y = x, \quad u(x, 0) = x^2.$$

In what domain is your solution determined by the initial data?

Problem 5 Solve the modified (damped) wave equation:

$$u_{tt} + 2u_t + u = u_{xx}$$

on the whole line with initial data $u(x, 0) = xe^{-x^2}$ and $u_t(x, 0) = 1$.
Hint: Consider $w(x, t) = e^t u(x, t)$.

Problem 6 Suppose f is a function of one variable that has continuous second derivative. Show that for any constants a and b, the function

$$u(x, y) = f(ax + by)$$

is a solution of the PDE

$$u_{xx}u_{yy} - u_{xy}^2 = 0$$

Problem 7 Give an example that shows why solutions of the wave equation $u_{tt} = u_{xx}$ do not necessarily satisfy the maximum principle (i.e., give an example of an explicit solution of the equation for which the maximum principle foes not hold).

Problem 8 Solve $u_x - yu_y + 2u = 1, u(x, 1) = 0$. In what domain in the plane is your solution determined?

Problem 9 Find the general solution $u(x, y)$ of the equation $u_x + u_{xy} = 1$.

Problem 10 Let $u(x, t)$ be the temperature in a rod of length L that satisfies the partial differential equation

$$u_t = ku_{xx} - ru, \quad (x, t) \in (0, L) \times (0, \infty),$$

where k and r are positive constants - this is related to the heat equation, but assumes that heat radiates out into the air along the rod - together with the initial condition

$$u(x, 0) = \phi(x), \quad x \in [0, L],$$

where ϕ satisfies $\phi(0) = \phi(L) = 0$ and $\phi(x) > 0$ for $x \in (0, L)$.

a) If u also satisfies the Dirichlet boundary conditions

$$u(0, t) = 0, \quad u(L, t) = 0$$

(so that the ends of the rod are held at temperature 0), show that the total "heat energy" in the rod at time t, which is given by

$$E(t) = \int_0^L u^2(x, t)dx,$$

is a strictly decreasing function of t.

b) Show that even if u satisfies Neumann boundary conditions

$$u_x(0,t) = u_x(L,t) = 0,$$

(so that the ends of the rod are insulated), it is still the case that $E(t)$ as defined above is a strictly decreasing function of t.

c) (Extra credit) Prove that in either a) or b), it must be the case that

$$\lim_{t\to\infty} E(t) = 0.$$

Problem 11 This problems concerns d'Alembert's solution to the initial-value problem for the wave equation $u_{tt} = c^2 u_{xx}$, together with initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

a) Show that if $f(x)$ and $g(x)$ are periodic functions with period $2L$ (so $f(x+2L) = f(x)$) for all x, and likewise for g), and if

$$\int_{-L}^{L} g(x)dx = 0,$$

then $u(x,t)$ is always periodic in x with period $2L$, in other words, $u(x+2L,t) = u(x,t)$, for all x and t.

b) Continuation of part a). With the periodicity assumptions of part a), show that $u(x,t)$ is also periodic in t. What is its period?

c) (Separate from parts a) and b).) Now suppose that $f(x)$ and $g(x)$, rather than being periodic, actually vanish outside of some finite interval, i.e., $f(x) = 0$ and $g(x) = 0$ for $|x| > R$. Show that

$$\lim_{t\to\infty} u(x,t)$$

is independent of x and give an expression for the limit in terms of f and/or g.