Name ______________________________

Math 312 - Section 002 - Midterm 1
Thursday, February 14, 2019, @ 10:30 AM - 11:50 AM

No form of cheating will be tolerated. You are expected to uphold the Code of Academic Integrity. I certify that all of the work on this test is my own.

Signature: ______________________

The use of calculators, computers and similar devices is neither necessary nor permitted during this exam. Correct answers without proper justification will not receive full credit. Clearly highlight your answers and the steps taken to arrive at them: illegible work will not be graded. You may use both sides of one 8 × 11 cheat-sheet.

OFFICIAL USE ONLY:

Problem	Points	Your score
1	25	
2	15	
3	20	
4	20	
5	20	
Total	100	

Problem 1 [25 points]

Consider the matrix $A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & -1 \\ 2 & 0 & -2 & 6 \end{bmatrix}$.

Part a. [5 points] Find an $A = LU$ decomposition (L lower triangular, U upper triangular).

The decomposition of the above matrix A into the row reduced echelon form $EA = R$ is given by

$$R = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}.$$

Part b. [5 points] Find a basis $\mathcal{B}$ for the column space of A, $C(A)$.

Part c. [5 points] Find a basis for the nullspace of A, $N(A)$.

Part d. [5 points] Find the complete solution to $A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ and $A\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

Part e. [5 points] Find the coordinates of $\vec{b} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ in the basis $\mathcal{B}$ you found in part b.

Problem 2 [15 points]

Construct a matrix with the following properties or explain why it is not possible:

Part a. [5 points] A 3 by 3 matrix with $C(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad N(A) = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$

Part b. [5 points] A matrix whose column space is spanned by $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and whose nullspace is spanned by $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 6 \end{bmatrix}$.

Part c. [5 points] A 2 by 2 matrix A such that the complete solution to the system $A\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is given by $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $(\alpha \in \mathbb{R})$.

Problem 3 [20 points]

Let V be the set of points (w, x, y, z) in $\mathbb{R}^4$ such that $w+x+y+z = 0$ and $2w+x-y-z = 0$.

Part a. [5 points] Find a basis for V.

Part b. [5 points] Find a matrix with four rows and four columns that satisfies $C(A) = V$.

Part c. [5 points] Let $\vec{b} = \begin{bmatrix} 4 \\ -6 \\ -1 \\ 3 \end{bmatrix}$. Can we ensure that, for any matrix A satisfying part b), the system $A\vec{x} = \vec{b}$ has infinite solutions? Explain your answer.

Part d. [5 points] Find the coordinates of $\vec{b}$ in the basis $\mathcal{V} = \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Problem 4 [20 points]

Part a. [10 points] Given the following bases for for the space of polynomials of degree at most 2, $\mathcal{P}_2$, find $P_{\mathcal{C}\leftarrow\mathcal{B}}$ the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

$$\mathcal{B} = \{x - 2, x^2 + 2x + 3, 3x^2\},$$
$$\mathcal{C} = \{x^2 + x + 1, x + 1, 1\}.$$

Part b. [5 points] Write the polynomial $1 + 3x - x^2$ in terms of the basis $\mathcal{B}$.

Part c. [5 points] Write the polynomial $1 + 3x - x^2$ in terms of the basis $\mathcal{C}$.

Problem 5 [20 points]

In each of the following cases, clearly mark the statement as **true** or **false**. Please also explain your answers in order to receive credit for this problem.

1. Let A be a 3 by 3 matrix. If the left-nullspace of A is given by the vectors $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfying $x_1 + x_2 + x_3 = 0$, then A cannot be invertible.

2. If A and B are matrices with rank 1, then so is $(A + B)/2$.

3. If P_{12} is the matrix that changes row 1 with row 2, then $P_{12}^2 = I$ (I denotes the identity matrix).

4. If a 4x3 matrix has 3 pivots, then $A\vec{x} = \vec{b}$ always has at least a solution.

5. The set of 3 by 3 matrices with the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in their column space is a vector space.

6. The set of points satisfying both $y = 3x + z$ and $z + x = 0$ is a subspace of $\mathbb{R}^3$.

7. The matrices A and A^T always have the same number of pivots.

8. If A is a change of basis matrix from a basis $\{\vec{v}_1, \vec{v}_2\}$ to a basis $\{\vec{u}_1, \vec{u}_2\}$, and B is a change of basis matrix from the basis $\{\vec{u}_1, \vec{u}_2\}$ to a basis $\{\vec{w}_1, \vec{w}_2\}$, then AB is a change of basis matrix from $\{\vec{v}_1, \vec{v}_2\}$ to $\{\vec{w}_1, \vec{w}_2\}$.

Extra space for work:

Extra space for work:

Extra space for work:

Extra space for work: