Name: ______________________________________ **PennID**: ________________

Math 425/AMCS 525
Midterm 1 - Practice Exam
February 7, 2019

Please *turn off and put away all electronic devices.* You are allowed to use an index card (3x5") with hand-written notes on both sides during this exam. No calculators, no books. Read the problems carefully. **Show all work** (answers without proper justification will not receive full credit). Be as organized as possible: illegible work will not be graded. Please sign and date the pledge below to comply with the Code of Academic Integrity. Don't forget to write your Name and PennID on the top of this page. Good luck!

#	Points possible	Your score
1	15	
2	15	
3	15	
4	15	
5	20	
6	20	
Total	100	

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination.

______________________________________ ________________

Signature Date

Problem 1 (15 pts): Find the general solution $u(x, y)$ of the equation

$$u_{xy} + 5u_y = 1.$$

Problem 2 (15 pts): Use the coordinate method to find $u(x,t)$ that solves

$$u_{xx} - 3u_{xt} + 2u_{tt} = 2x + 2t.$$

Hint: Factor the operator first (then deduce the appropriate change of variables).

Problem 3 (15 pts): Solve

$$u_x - yu_y = 0$$
$$u(x, 1) = 2e^x$$

and find the region of the xy plane in which the solution is uniquely determined.

Problem 4 (15 pts) Consider the Dirichlet problem for the wave equation:

$$u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \; t > 0$$
$$u(x, 0) = \phi(x)$$
$$u_t(x, 0) = \psi(x).$$

(a) Derive the d'Alembert formula.

(b) Show that if ϕ and ψ are even functions, then u is also even in x for all t.

Problem 5 (20 pts) Suppose $u(x,t)$ satisfies the diffusion equation

$$u_t = 2u_{xx}, \qquad \text{for } 0 < x < 1, t > 0,$$

subject to the follwing initial and boundary conditions

$$u(x,0) = \sin(\pi x),$$
$$u(0,t) = u(1,t) = 0.$$

(a) Use the maximum principle to show that there is at most one solution $u(x,t)$.

(b) Show that $u(x,t) > 0$ at all interior points $0 < x < 1$, $0 < t < \infty$. (*Hint:* You may use the strong maximum principle.)

(c) For each $t > 0$, define $M(t)$ = the maximum of $u(x,t)$ over $0 \leq x \leq 1$. Show that $M(t)$ is nonincreasing function of t. (*Hint:* Let the maximum occur at the point $X(t)$ so that $M(t) = u(X(t),t)$. Differentiate $M(t)$ assuming that $X(t)$ is differentiable.)

Problem 6 (20 pts) Suppose $u(x,t)$ satisfies the following problem:

$$\begin{aligned} &u_t = u_{xx} - 3u, && \text{for } 0 < x < 1, t > 0, \\ &u(x,0) = x(1-x), && \text{for } 0 < x < 1 \\ &u(0,t) = u(1,t) = 0, && \text{for } t > 0. \end{aligned}$$

Define the total heat energy by

$$E(t) = \frac{1}{2}\int_0^1 u^2(x,t)dx.$$

(a) Show that $E(t)$ is a non-increasing function of t.

(b) Show that

$$\lim_{t\to\infty} E(t) = 0.$$

(*Hint for part (b):* Show that $\frac{dE}{dt} \leq -6E$, then solve this differential inequality).

Extra paper

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