Name ____

Math 312 - Section 002 - Midterm 1 (Practice exam)
Thursday, February 14, 2019, @ 10:30 AM - 11:50 AM

No form of cheating will be tolerated. You are expected to uphold the Code of Academic Integrity. I certify that all of the work on this test is my own.

Signature: ______________________________

The use of calculators, computers and similar devices is neither necessary nor permitted during this exam. Correct answers without proper justification will not receive full credit. Clearly highlight your answers and the steps taken to arrive at them: illegible work will not be graded. You may use both sides of one 8 $\times$ 11 cheat-sheet.

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Problem	Points	Your score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Problem 1 [20 points]

Consider the matrix $A = \begin{bmatrix} 1 & 3 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 \\ 2 & 6 & 4 & 4 & 3 \\ 0 & 0 & 2 & 2 & 3 \end{bmatrix}$.

Part a. Find the matrix R, the RREF (row reduced echelon form) of A.

$$\begin{bmatrix} 1 & 3 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 \\ 2 & 6 & 4 & 4 & 3 \\ 0 & 0 & 2 & 2 & 3 \end{bmatrix} \xrightarrow{R3' = R3 - 2R1} \begin{bmatrix} 1 & 3 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & 2 & 1 \\ 0 & 0 & 2 & 2 & 3 \end{bmatrix} \xrightarrow[R4' = R4 + 2R2]{R3'' = R3' + 2R2}$$

$$\to \begin{bmatrix} 1 & 3 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \xrightarrow[R1' = R1 - R3'']{R4'' = R4' - 3R3''} \begin{bmatrix} 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R2' = -R2]{R1'' = R1' + R2}$$

$$\to \begin{bmatrix} \boxed{1} & 3 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Part b. Find the matrix E that transforms A into R, i.e., $EA = R$.

$$E = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

order important!

$$= \begin{bmatrix} 3 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -2 & 2 & 1 & 0 \\ 6 & -4 & -3 & 1 \end{bmatrix} \leftarrow \text{check } EA = R$$

Part c. Find the complete solution to $A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}$.

$\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}$: $A\vec{x} = \vec{b} \Leftrightarrow \underset{\underset{EA = R}{\uparrow}}{R\vec{x} = E\vec{b}}$

$$\begin{bmatrix} \boxed{1} & 3 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -2 & 2 & 1 & 0 \\ 6 & -4 & -3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\left.\begin{matrix} x_2 = \alpha \\ x_4 = \beta \end{matrix}\right\} \to \begin{matrix} x_1 = -1 - 3\alpha \\ x_3 = -\beta \\ x_5 = 1 \end{matrix} \;\Bigg|\; \to \; \vec{x} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

Part d. Find a basis for each one of the four fundamental subspaces of A.

Basis for $N(A) = \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ (in $\mathbb{R}^5$) (from free variables)

Basis for $C(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \end{bmatrix} \right\}$ (in $\mathbb{R}^4$) (pivot columns of $\underline{\underline{A}}$)

Basis for $C(A^T) = \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ (in $\mathbb{R}^5$)

Basis for $N(A^T) = \left\{ \begin{bmatrix} 6 \\ -4 \\ -3 \\ 1 \end{bmatrix} \right\}$ (in $\mathbb{R}^4$) (last row of E)

- check dimensions:

$n = 5 = 3 + 2 = \dim C(A) + \dim N(A)$.

$m = 4 = 3 + 1 = \dim C(A^T) + \dim N(A^T)$.

Problem 2 [20 points]

Forward elimination changes $A\vec{x} = \vec{b}$ to a row reduced echelon form $R\vec{x} = \vec{d}$. The complete solution is

$$\vec{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}, \quad \alpha, \beta \in \mathbb{R}.$$

Part a. What is the 3 by 3 reduced row echelon matrix R and what is $\vec{d}$?

R 3×3, $\dim N(R) = 2$ (two parameters) $\rightarrow$ 1 pivot.

$$\left.\begin{aligned} x_1 &= 4 + 2\alpha + 5\beta \\ x_2, x_3 &\text{ free} \end{aligned}\right\} \quad R = \begin{bmatrix} 1 & -2 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \vec{d} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

Part b. If the process of elimination subtracted 3 times row 1 from row 2 and then 5 times row 1 from row 3, what matrix connects R and $\vec{d}$ to the original A and $\vec{b}$?

$$A \xrightarrow[R3' = R3 - 5R1]{R2' = R2 - 3R1} R$$

$$\Downarrow$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}, \quad EA = R$$

$$A = E^{-1}R, \quad E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$$

$$\vec{b} = E^{-1}\vec{d}$$

Part c. Find the LU decomposition of A. (Hint: use part b)

Notice that $L = E^{-1}$ since it is lower triangular with 1's on the diagonal.

So $A = LU$, $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$, $U = R = \begin{bmatrix} 1 & -2 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Problem 3 [20 points]

Let $\mathbf{u}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}2\\1\\0\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix}2\\0\\2\end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix}0\\2\\1\end{bmatrix}$.

Part a. Find a basis for $V = \{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4, \vec{u}_5\}$. What is the dimension of V?

$$\begin{bmatrix}1 & 2 & 1 & 2 & 0\\ 0 & 1 & 1 & 0 & 2\\ 1 & 0 & 1 & 2 & 1\end{bmatrix} \rightarrow \begin{bmatrix}1 & 2 & 1 & 2 & 0\\ 0 & 1 & 1 & 0 & 2\\ 0 & -2 & 0 & 0 & 1\end{bmatrix} \rightarrow \begin{bmatrix}\boxed{1} & 2 & 1 & 2 & 0\\ 0 & \boxed{1} & 1 & 0 & 2\\ 0 & 0 & \boxed{2} & 0 & 5\end{bmatrix}$$

Three pivots $\Rightarrow$ 3 l.i. vectors $\Rightarrow$ dim $V = 3$.

Since $V \subseteq \mathbb{R}^3$, dim $V = 3 \Rightarrow V = \mathbb{R}^3$.

A basis for $V = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$, another $= \left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}, \ldots$

Part b. Show that $\{1, 1-x, (1-x)^2\}$ is a basis of $\mathcal{P}_2$, the vector space of polynomials of degree at most 2.

$\{1, 1-x, (1-x)^2\}$ basis of $\mathcal{P}_2$ if
1) They are l.i.
2) They span P_2.

1) $c_1 + c_2(1-x) + c_3(1-x)^2 = 0 \Rightarrow c_1 = c_2 = c_3 = 0$?

$\Leftrightarrow c_1 + c_2 - c_2 x + c_3(1 + x^2 - 2x) = 0 \Leftrightarrow \left.\begin{aligned} c_1 + c_2 + c_3 &= 0\\ -c_2 - 2c_3 &= 0\\ c_3 &= 0\end{aligned}\right\} \Leftrightarrow \begin{bmatrix}1 & 1 & 1\\ 0 & -1 & -2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \Leftrightarrow$

$\Leftrightarrow c_1 = c_2 = c_3 = 0$. So they are l.i.

2) Any $p(x) = a + bx + x^2 c$ can be writte as a lin. comb. of them?

Yes, $\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\ 0 & -1 & -2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$ always has a solution (becouse three pivots).

Problem 4 [20 points]

Part a. Given the following bases for for the space of polynomials of degree at most 2, $\mathcal{P}_2$, find $P_{\mathcal{C}\leftarrow\mathcal{B}}$ the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

$$\mathcal{B} = \{5x^2 - x - 6, 6x^2 + 8x + 9, 3x^2 + 11x + 8\},$$
$$\mathcal{C} = \{x^2 + x + 1, x + 1, 1\}.$$

$$M_B = \begin{bmatrix} -6 & 9 & 8 \\ -1 & 8 & 11 \\ 5 & 6 & 3 \end{bmatrix}, \quad M_C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad P_{C\leftarrow B} = M_C^{-1} M_B$$

$$P_{C\leftarrow B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -6 & 9 & 8 \\ -1 & 8 & 11 \\ 5 & 6 & 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & -6 & 9 & 8 \\ 1 & 1 & 0 & -1 & 8 & 11 \\ 1 & 0 & 0 & 5 & 6 & 3 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & -6 & 9 & 8 \\ 0 & 0 & -1 & 5 & -1 & 3 \\ 0 & -1 & -1 & 11 & -3 & -5 \end{array}\right] \to$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 6 & 3 \\ 0 & 0 & 1 & -5 & 1 & -3 \\ 0 & -1 & 0 & 6 & -2 & -8 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 6 & 3 \\ 0 & 1 & 0 & -6 & 2 & 8 \\ 0 & 0 & 1 & -5 & 1 & -3 \end{array}\right], \quad P_{C\leftarrow B} = \begin{bmatrix} 5 & 6 & 3 \\ -6 & 2 & 8 \\ -5 & 1 & -3 \end{bmatrix}$$

Part b. Write the polynomial $(5x^2 - x - 6) + (6x^2 + 8x + 9)$ in terms of the basis $\mathcal{C}$.

We are given $p(x) = (5x^2 - x - 6) + (6x^2 + 8x + 9) \Rightarrow [p]_B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow$

$$\Rightarrow [p]_C = \begin{bmatrix} 5 & 6 & 3 \\ -6 & 2 & 8 \\ -5 & 1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \\ -4 \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(x) = 11(x^2 + x + 1) - 4(x + 1) - 4.$$

Problem 5 [20 points]

In each of the following cases, clearly mark the statement as **true** or **false**. Please also explain your answers in order to receive credit for this problem.

1. Any square matrix A such that $N(A) = \{\vec{0}\}$ always has an inverse.

True: $N(A) = \{\vec{0}\} \Rightarrow \dim C(A) = n \to$ full set of pivots //.

2. Let A, B, C be invertible matrices. Then, the product ABC is always invertible.

True: A^{-1}, B^{-1}, C^{-1} exists, so $C^{-1}B^{-1}A^{-1}$ is well defined.
We can check that $C^{-1}B^{-1}A^{-1}ABC = ABCC^{-1}B^{-1}A^{-1} = I$, thus
$C^{-1}B^{-1}C^{-1}$ is the inverse of ABC //.

3. Consider an elementary matrix E that adds two times row 1 to row 2. Then the (2,1) entry of E^{100} is 2^{100}.

False: Counterexample: $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$.

Indeed, $[E^{100}]_{2,1} = 2 \cdot 100 = 200$

4. If a 4x3 matrix has 3 pivots, then $A\vec{x} = \vec{b}$ always has at least a solution.

False: Counterexample:
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ does not have a solution //.

5. Let V be the set of 2 by 2 matrices. Then, the set of symmetric 2 by 2 matrices is a subspace of V.

True: Denote $\mathcal{U} \equiv$ set of 2x2 symmetric matrices.

1) $A, B \in \mathcal{U} \Rightarrow A + B \in \mathcal{U}$? $\begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_2 & b_3 \end{bmatrix} = \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_2+b_2 & a_3+b_3 \end{bmatrix} \in \mathcal{U}$ //.

2) $A \in \mathcal{U}$, $\lambda \in \mathbb{R}$ $\Rightarrow \lambda A \in \mathcal{U}$? $\lambda\begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} = \begin{bmatrix} \lambda a_1 & \lambda a_2 \\ \lambda a_2 & \lambda a_3 \end{bmatrix} \in \mathcal{U}$ // (*)

6. Consider the set of vectors $\vec{x}$ such that $A\vec{y} = \vec{x}$ always has a solution. Is that a subspace?

True: Indeed, this is the definition of $C(A)$.

[redacted]

7. Let $A = LU$ (L lower triangular with ones on the diagonal, U upper triangular). Let $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ be the columns of A and $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$ be the columns of U. Then, if $\vec{u}_3 = 2\vec{u}_2 - \vec{u}_1$, we also have that $\vec{a}_3 = 2\vec{a}_2 - \vec{a}_1$.

True: Notice that $\vec{a}_i = L\vec{u}_i$, so we only need to multiply by L.

8. If A is a change of basis matrix from a basis $\{\vec{v}_1, \vec{v}_2\}$ to a basis $\{\vec{u}_1, \vec{u}_2\}$, and B is a change of basis matrix from the basis $\{\vec{u}_1, \vec{u}_2\}$ to a basis $\{\vec{w}_1, \vec{w}_2\}$, then AB is a change of basis matrix from $\{\vec{v}_1, \vec{v}_2\}$ to $\{\vec{w}_1, \vec{w}_2\}$.

False:

$$\left.\begin{array}{l} \vec{b}|_u = A\,\vec{b}|_v \\ \vec{b}|_w = B\,\vec{b}|_u \end{array}\right\} \Rightarrow \vec{b}|_w = \underline{\underline{BA}}\,\vec{b}|_v\text{, so not } AB,$$

(unless $AB = BA$)

(*) For any size of the matrices:

1) $A, B \in \mathcal{U} \Rightarrow A + B \in \mathcal{U}$? $(A+B)^T = A^T + B^T = A + B$ ✓.

2) $\left.\begin{array}{l} A \in \mathcal{U} \\ \lambda \in \mathbb{R} \end{array}\right\} \Rightarrow \lambda A \in \mathcal{U}$? $(\lambda A)^T = \lambda A^T = \lambda A$ ✓.