$\boxed{\text{PRACTICE EXAM}}$: Solutions (Math 425)

$\boxed{P1}$ $u_{xy} + 5u_y = 1$.

Sol.: Denote $v(x,y) = u_y(x,y)$.

Then,

$$v_x + 5v = 1 \Rightarrow v(x,y) = e^{-5x}\int e^{5x}\,dx = \frac{1}{5} + c(y)e^{-5x}.$$

Finally,

$$u_y(x,y) = \frac{1}{5} + c(y)e^{-5x} \Rightarrow \boxed{u(x,y) = \frac{1}{5}y + e^{-5x}f(y) + g(x)}$$

where f, g are two functions (of one variable).

$\boxed{P2}$ $u_{xx} - 3u_{xt} + 2u_{tt} = 2x + 2t$

[Solve by coordinate method]

<u>Sol:</u>

We first factor the operator:

$$\partial_{xx} - 3\partial_{xt} + 2\partial_{tt} \overset{(*)}{=} (\partial_x - \partial_t)(\partial_x - 2\partial_t)$$, that is, the equation rewrites as

$$(\partial_x - \partial_t)(\partial_x - 2\partial_t)u = 2x + 2t.$$

Notice that each of the new first-order operators are of transport type. We introduce the following change of variables:

$$(**) \quad \left.\begin{array}{l} \eta = 2x + t \\ \xi = x + t \end{array}\right\} \Longrightarrow \left.\begin{array}{l} x = \eta - \xi \\ t = 2\xi \end{array}\right\}$$

In these new variable we have that

- $\partial_x - \partial_t = \partial_\eta \dfrac{\partial \eta}{\partial x} + \partial_\xi \dfrac{\partial \xi}{\partial x} - \partial_\eta \dfrac{\partial \eta}{\partial t} - \partial_\xi \dfrac{\partial \xi}{\partial t} =$

$$= 2\partial_\eta + \cancel{\partial_\xi} - \partial_\eta - \cancel{\partial_\xi} = \partial_\eta,$$

- $\partial_x - 2\partial_t = 2\cancel{\partial_\eta} + \partial_\xi - 2\cancel{\partial_\eta} - 2\partial_\xi = -\partial_\xi.$

thus the PDE becomes

$$-\partial_{\eta\xi} u = 2\xi \Rightarrow \partial_\eta u = -\int 2\xi \, d\xi = -\xi^2 + c(\eta) \Rightarrow$$

$$\Rightarrow u(\eta,\xi) = -\xi^2 \eta + f(\eta) + g(\xi).$$

- Going back to (x,t):

$$\boxed{u(x,t) = -(x+t)^2(2x+t) + f(2x+t) + g(x+t).}$$

Remark: We had an inhomogeneous linear equation. The solution is equal to the solution of the homogeneous equation plus a particular solution.

↳ Again, one can check the result by plugging u into the PDE.

(*) We can always write $(\partial_x + a\partial_t)(\partial_x + b\partial_t) = \partial_{xx} - 3\partial_{xt} + 2\partial_{tt}$, and obtain a, b from there: $\begin{matrix} a+b=-3 \\ ab=2 \end{matrix} \rightarrow \begin{matrix} a=-1 \\ b=-2. \end{matrix}$

(**) This is the change of variables we did in transport equations.

Basically, for $(a\partial_x + b\partial_y)u$, we notice that this is

$(a\partial_x + b\partial_y)u = (a,b)\cdot\nabla u \equiv$ derivative in direction (a,b).

That is, we are keeping the $(-b,a)$ direction fixed.
(the orthogonal one)

↳ So choose $\eta = -bx + ay$ as new variable.

→ We can choose the other one as we prefer.

↳ But here, since we also want to solve a second transport equation, we have our new two variables determined.

$\boxed{P3}$ $\left.\begin{aligned} &u_x - y u_y = 0 \\ &u(x,1) = 2e^x \end{aligned}\right\}$

Sol: Let $(x(t), y(t))$ be the characteristic curves of the problem (parametrized by t).

Then,

$$\frac{d}{dt}\Big(u(x(t), y(t))\Big) = u_x(x(t), y(t))\, x'(t) + u_y(x(t), y(t))\, y'(t).$$

Defining $x(t), y(t)$ by

$$\left.\begin{aligned} x'(t) &= 1 \\ y'(t) &= -y \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x(t) &= t + x(0) \\ y(t) &= y(0)\, e^{-t} \end{aligned}\right\}$$, we obtain that

$$\left.\begin{aligned} &u'(t) = 0 \\ &u(0) = u(x(0), y(0)) \end{aligned}\right\}$$

Let's choose $x(0) = c$, $y(0) = 1 \rightarrow \left| \begin{aligned} x(t) &= c + t \\ y(t) &= e^{-t} \end{aligned}\right.$,

then

$$\left.\begin{aligned} &u'(t) = 0 \\ &u(0) = u(c,1) = 2e^{c} = 2e^{c} \end{aligned}\right\} \Rightarrow u(t) = 2e^{c}.$$

Going back,

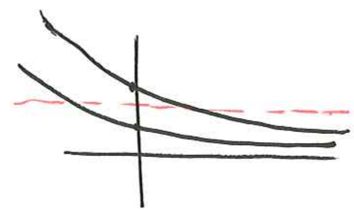
$$\left.\begin{array}{l} x = c + t \\ y = e^{-t} \end{array}\right\} \to e^{x} = e^{c} e^{t} \Rightarrow e^{c} = e^{x} e^{-t} \Rightarrow e^{c} = e^{x} y,$$

thus,

$$\boxed{u(x,y) = 2 e^{x} y}$$

- Region where uniquely determined:

$$\left.\begin{array}{l} x = c + t \\ y = e^{-t} \end{array}\right\} \to y = e^{c} e^{-x} \rightsquigarrow$$



The initial data crosses all the characteristics curves.
These fill (only) the half-plane $y > 0$:

$$\{(x,y) \in \mathbb{R}^2 : y > 0\}.$$

• Also:

Parametrize the characteristic curves as $(x, y(x))$:

$y' = -y \Rightarrow y(x) = c e^{-x}$.

Therefore,

$\frac{d}{dx}\left(u(x, y(x))\right) = 0 \Rightarrow u(x, y(x)) = u(0, y(0)) \Rightarrow$

$\Rightarrow u(x, c e^{-x}) = u(0, c) = f(c)$.

That is,

$\| \; u(x, y) = f(y e^{x})$ (general solution).

Initial data:

(since $e^x > 0$)

$u(x, 1) = 2e^x = f(e^x) \Rightarrow f(s) = 2s$ (for $\underline{s > 0}$)

In conclusion,

$\| \; u(x, y) = f(y e^x) = 2 y e^x \; \|$ in the region $\{(x, y) \in \mathbb{R}^2 : y > 0\}$.

$\boxed{P4}$

a) See theory.

b) We assume that $\left.\begin{array}{l}\phi(x)=\phi(-x)\\ \psi(x)=\phi(-x)\end{array}\right\}$ (even functions).

We want to show that $u(x,t)=u(-x,t)\ \forall t>0$.

$$\rightarrow\ u(x,t)=\frac{1}{2}\Big(\phi(x+ct)+\phi(x-ct)\Big)+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds.$$

$$\rightarrow\ u(-x,t)=\frac{1}{2}\Big(\phi(-x+ct)+\phi(-x-ct)\Big)+\frac{1}{2c}\int_{-x-ct}^{-x+ct}\psi(s)\,ds=$$

$$=\frac{1}{2}\Big(\phi(x-ct)+\phi(x+ct)\Big)+\frac{-1}{2c}\int_{x+ct}^{x-ct}\psi(-z)\,dz=$$

$(\phi(x)=\phi(-x))$ $\qquad$ $(z=-s)$

$$=\frac{1}{2}\Big(\phi(x+ct)+\phi(x-ct)\Big)+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(z)\,dz=u(x,t)\,/\!/.$$

$\psi(-z)=\psi(z)$

$\boxed{P5}$

a) See theory.

b)

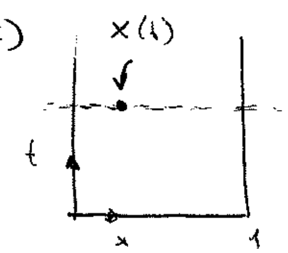
The strong maximum principle guarantees that the maximum of the solution is <u>only</u> attained on the boundary $x = 0$ or $x = 1$ or $t = 0$.

Analogously with the strong minimum principle.

Notice that the minimum value of the boundary and initial data is zero:

$$u(x,t) > \min_{\substack{x=0 \\ \text{or } x=1 \\ \text{or } t=0}} \{u(x,t)\} = \min_{0 \le x \le 1} \{0, \sin(\pi x)\} = 0 \;/\!/.$$

c)



$$M(t) = \max_{0 \le x \le 1} \{u(x,t)\} = u(X(t),t).$$

Then,
$$\left.\begin{aligned} u_x(X(t),t) &= 0 \\ u_{xx}(X(t),t) &\le 0 \end{aligned}\right\} (*)$$

Therefore,

$$M'(t) = \underbrace{u_x(X(t),t)}_{=0} X'(t) + u_t(X(t),t) = 2u_{xx}(X(t),t) \le 0 \;/\!/.$$

(*) From part b), we know that in the interior points

$u(x,t) > 0$.

So, $M(t) = \max_{0 \le x \le 1} \{u(x,t)\} = \max_{0 < x < 1} \{u(x,t)\} = u(x(t), t)$.

That is, the maximum $M(t)$ occurs at an interior point, so the optimality conditions ($u_x = 0$, $u_{xx} < 0$) must be satisfied.

$\boxed{P6}$

a) We have to show that $E'(t) \leq 0$.

$$E'(t) = \frac{1}{2}\frac{d}{dt}\int_0^1 u^2(x,t)\,dx = \int_0^1 u(x,t)\,u_t(x,t)\,dx =$$

$$= \int_0^1 u(x,t)\,u_{xx}(x,t)\,dx - 3\int_0^1 (u(x,t))^2\,dx =$$

$$= -\int_0^1 (u_x(x,t))^2\,dx + \underbrace{u(x,t)\,u_x(x,t)\Big]_{x=0}^{x=1}}_{=0} - 3\int_0^1 (u(x,t))^2 =$$

$$= -\int_0^1 \left((u_x(x,t))^2 + 3\,(u(x,t))^2\right)dx \leq 0$$

! We have used that $u(0,t) = u(1,t) = 0$.

b) From part a) we have that

$$E'(t) = -\int_0^1 (u_x(x,t))^2 - 3\int_0^1 (u(x,t))^2\,dx \leq -3\int_0^1 (u(x,t))^2\,dx =$$

$$= -6\,\frac{1}{2}\int_0^1 (u(x,t))^2\,dx = -6\,E(t).$$

That is,

$$E'(t) \leq -6\,E(t) \Rightarrow E(t) \leq E(0)\,e^{-6t} \Rightarrow$$

$$\lim_{t\to+\infty} E(t) \leq E(0) \lim_{t\to+\infty} e^{-6t} = 0 \Rightarrow \lim_{t\to+\infty} E(t) = 0 .$$

$$\left[(*)\ E(t) \geq 0 \text{ from definition; } E(0) = \frac{1}{2}\int_0^1 x^2(1-x)^2\,dx < +\infty \right].$$