| MATH 425 LECTURE 8 | : Review for Midterm 1.

- P2 of practice exam → see solution there.

- <u>P4</u> (Problems from previous exams)

$$\left.\begin{array}{l} y u_x + u_y = x \\ u(x,0) = x^2 \end{array}\right\}$$

Let $(x(t), y(t))$ be a parametrization of the characteristic curves, defined by

$$\left.\begin{array}{l} x'(t) = y \\ y'(t) = 1 \end{array}\right\} \longrightarrow \begin{array}{l} x'(t) = t + y(0) \to x(t) = \frac{t^2}{2} + y(0)t + x(0), \\ y(t) = t + y(0). \end{array}$$

Then,

$$\frac{d}{dt}\Big(u(x(t), y(t))\Big) = x(t) = \frac{t^2}{2} + y(0)t + x(0).$$

That is,

$$\left.\begin{array}{l} u'(t) = t^2/2 + y(0)t + x(0) \\ u(0) = u(x(0), y(0)) \underset{\uparrow}{=} u(c,0) = c^2 \end{array}\right\}$$

choose $x(0) = c$, $y(0) = 0$

$u'(t) = t^2/2 + c \Rightarrow u(t) = t^3/6 + ct + c_1,$

$u(0) = c_1 = c^2$

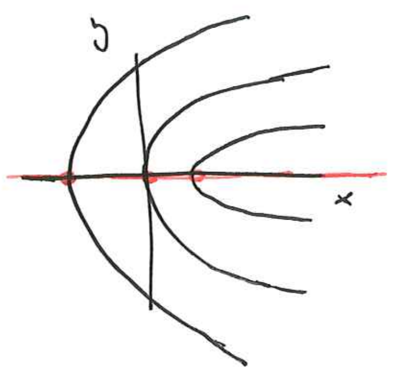
$\Rightarrow u(t) = t^3/6 + ct + c^2.$

Finally, substitute back: $\left.\begin{array}{l} x = t^2/2 + c \\ y = t \end{array}\right\} \to \left\{\begin{array}{l} c = x - y^2/2 \\ t = y \end{array}\right.$

$$u(x,y) = y^3/6 + (x - y^2/2)y + (x - y^2/2)^2$$

- Region: Notice that for all (x,y) we have t, c well-defined (and viceversa).

Also, the characteristic ~~lines~~ curves are $x = y^2/2 + c$:



so any point $(x,y) \in \mathbb{R}^2$ is in a characteristic curve, and the initial data crosses all these curves.

$\boxed{P2}$ (previous exams)

$$\left.\begin{array}{ll} u_t = k u_{xx}, & 0 \le x \le L \\ u(x,0) = \phi(x). & t \ge 0 \end{array}\right\} \text{ with } \phi(0) = \phi(L) = 0 \text{ and}$$

$\phi(x) > 0$ for $0 < x < L$.

a) If in addition

$u_x(0,t) = 0 = u_x(L,t)$ for $t \ge 0$,

show that $A(t) = \frac{1}{L}\int_0^L u(x,t)\,dx$ is constant.

<u>Sol:</u>

$$\frac{d}{dt}(A(t)) \equiv A'(t) = \frac{1}{L}\int_0^L u_t(x,t)\,dx = \frac{k}{L}\int_0^L u_{xx}(x,t)\,dx =$$

$$= \frac{k}{L}\Big(u_x(L,t) - u_x(0,t)\Big) = 0 \Rightarrow A(t) = A(0) \equiv \text{cte}_{/\!/}.$$

b) If now $u(0,t) = 0 = u(L,t)$, show that

$u(x,t) \ge 0$ for $0 < x < L$, $t > 0$.

<u>Sol:</u> By the Minimum Principle for the heat equation,

$$u(x,t) \ge \min_{\substack{0 \le x \le L \\ t \ge 0}} \{u(0,t), u(L,t), u(x,0)\} =$$

$$= \min_{\substack{0 \le x \le L \\ t \ge 0}} \{0, 0, \phi(x)\} = 0 \quad (\text{since } \phi(x) \ge 0).$$

c) Show, with the conditions of part b), that $A(t)$ is a non-increasing function of t.

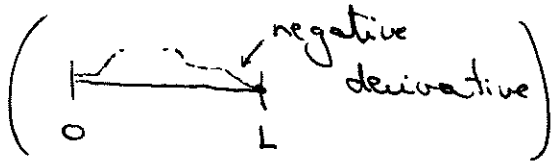
<u>Sol.</u>:

From a),

$$A'(t) = \frac{k}{L}\left(u_x(L,t) - u_x(0,t)\right).$$

But now, notice that $u_x(L,t) \le 0$, $u_x(0,t) \ge 0$, which proves $A'(t) \le 0$. Indeed:

$$\left.\begin{array}{l} u(L,t) = 0 \\ u(x,t) \ge 0 \end{array}\right\} \Rightarrow u_x(L,t) \le 0$$



$$\left.\begin{array}{l} u(0,t) = 0 \\ u(x,t) \ge 0 \end{array}\right\} \Rightarrow u_x(0,t) \ge 0.$$

$\boxed{P5}$

$$\left.\begin{aligned} &u_{tt} + 2u_t + u = u_{xx} \\ &u(x,0) = x e^{-x^2} \\ &u_t(x,0) = 1 \end{aligned}\right\} \quad \text{Hint: } w(x,t) = e^t u(x,t).$$

Sol:

$$w_x = e^t u_x \to w_{xx} = e^t u_{xx}$$

$$w_t = e^t u + e^t u_t \to w_{tt} = e^t u + 2e^t u_t + e^t u_{tt}.$$

Therefore,

$$u + 2u_t + u_{tt} = e^{-t} w_{tt} = u_{xx} = e^{-t} w_{xx} \Rightarrow \{e^{-t} > 0\}$$

$$\Rightarrow \| \ w_{tt} = w_{xx}.$$

Initial conditions: $w(x,0) = u(x,0) = x e^{-x^2}.$

$$w_t(x,0) = u(x,0) + u_t(x,0) =$$

$$= x e^{-x^2} + 1$$

Thus, using D'Alembert formula:

$$w(x,t) = \frac{1}{2}\left((x+t) e^{-(x+t)^2} + (x-t) e^{-(x-t)^2} \right) + \frac{1}{2}\int_{x-t}^{x+t} (s e^{-s^2} + 1)\, ds = \ldots$$

Finally, $\| \ u(x,t) = e^{-t} w(x,t) \ \|$.

$\boxed{P10}$ $\left.\begin{array}{ll} u_t = k u_{xx} - ru, & 0 \leq x \leq L \\ u(x,0) = \phi(x), & t \geq 0 \end{array}\right\}$ with $\begin{cases} \phi(x) > 0 & \text{for } x \in (0,L) \\ \phi(0) = \phi(L) = 0 \end{cases}$

a) If $u(0,t) = 0 = u(L,t)$ for $t \geq 0$, show that

$E(t) = \int_0^L u^2(x,t)\,dx$ is strictly decreasing in time.

Sol:

$$E'(t) = 2\int_0^L u(x,t)\,u_t(x,t)\,dx = 2k\int_0^L u(x,t)\,u_{xx}(x,t)\,dx +$$

$$- 2r\int_0^L (u(x,t))^2\,dx =$$

$$= -2k\int_0^L (u_x(x,t))^2\,dx - 2r\int_0^L (u(x,t))^2\,dx + \underbrace{\cancel{2k\,u(x,t)\,u_x(x,t)\Big]_{x=0}^{x=L}}}_{=0} \Rightarrow$$

$\Rightarrow E'(t) \leq 0$.

→ But we need to show $E'(t) < 0$ for all $t \geq 0$.

We have that

$$E'(t) \leq -2r\int_0^L (u(x,t))^2\,dx$$

Remark: As I commented in class, here we cannot conclude that $E'(t) < 0$ in the following simple way:

By contradiction, assume $\int_0^L (u(x,t))^2\,dx = 0$. Then,

$u(x,t) = 0$ for $0 \le x \le L$.

But this is not a contradiction right now (unless $t = 0$).

→ So we need indeed a maximum principle.

It turns out that it is quite simple to obtain it:

Let's do the change $w(x,t) = e^{rt}u(x,t)$. Then, one can check that

$$\left.\begin{aligned} &w_t = k\,w_{xx} \\ &w(x,0) = u(x,0) = \phi(x) \\ &w(0,t) = w(L,t) = 0 \end{aligned}\right\}$$ so w satisfies the strong ~~maximum~~ minimum principle, and hence,

$$\boxed{w(x,t) > 0 \text{ for all } 0 < x < L,\ t > 0}.$$

Therefore, $u(x,t) = e^{-rt}w(x,t) > 0$ for all $0 < x < L,\ t > 0$.

Now we can really conclude that

$$\int_0^L (u(x,t))^2\,dx > 0 \quad \text{for all } t \geq 0, \text{ and thus } \underline{\underline{E'(t) < 0}}\,.$$

b) Follows as a).

c) $E'(t) \leq -2r \int_0^L (u(x,t))^2\,dx = -2r\,E(t) \Rightarrow$

$\Rightarrow E(t) \leq E(0)\,e^{-2rt} \Rightarrow \lim_{t\to+\infty} E(t) \equiv 0.$

Since $E(t) \geq 0$, we have that $\lim_{t\to 0} E(t) = 0$ //.

$\boxed{\text{P11}}$

a) Check that $u(x+2L,t) = u(x,t)$ by writing D'Alembert formula.

b) Same as a), with time period $2L/c$.

c) Take $\lim_{t\to+\infty}$ in D'Alembert formula.

($\rightarrow$ see solution in Dennis DeTurk website: Midterm 1, March 2, 2010).