MATH 425 LECTURE 7	: The Heat Equation (II) (The solution)

<u>Exercise</u>: Show that solutions to the following IVBP,

$$\left.\begin{array}{ll} u_t - k u_{xx} = 0 , & 0 \le x \le L \\ u(x,0) = \phi(x) , & t > 0 \\ u(0,t) = 0 = u(L,t), & \end{array}\right\} [1]$$

are stable in the L^2 sense with respect to the data ϕ.

<u>Solution</u>:

Let u_1 solve [1] with data ϕ_1, and u_2 solve [2] with ϕ_2.

Then, the difference $v = u_1 - u_2$ solves

$$\left.\begin{array}{l} v_t - k v_{xx} = 0 \\ v(x,0) = \phi_1(x) - \phi_2(x) \\ v(0,t) = 0 = v(L,t) \end{array}\right\}$$

Multiplying by v, integrating over $[0,L]$ yields that

$$\frac{1}{2}\frac{d}{dt}\int_0^L (v(x,t))^2\,dx - k\int_0^L v(x,t)\,v_{xx}(x,t)\,dx = 0 \quad \xrightarrow{\text{Integration by parts.}}$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\int_0^L (v(x,t))^2\,dx + k\int_0^L (v_x(x,t))^2\,dx - \underbrace{k\, v_x(x,t)\,v(x,t)\Big|_{x=0}^{x=L}}_{=0} = 0$$

So we have that $\frac{d}{dt}\int_0^L (v(x,t))^2\,dx = -k\int_0^L (v_x(x,t))^2\,dx \leq 0 \Rightarrow$

$$\Rightarrow \frac{d}{dt}\int_0^L (v(x,t))^2\,dx \leq 0 \Rightarrow \int_0^L (v(x,t))^2\,dx \leq \int_0^L (v(x,0))^2\,dx \Rightarrow$$

$$\Rightarrow \int_0^L (u_1(x,t)-u_2(x,t))^2\,dx \leq \int_0^L (\phi_1(x)-\phi_2(x))^2\,dx \quad \text{for all } t\geq 0.$$

2.4) Solution to the heat equation.

Our goal now is to solve the homogeneous heat equation on the whole line for a general initial condition $\phi(x)$. That is, to solve

$$\left.\begin{array}{ll} u_t - k u_{xx} = 0, & -\infty < x < \infty \\ u(x,0) = \phi(x) & t>0 \end{array}\right\} \text{[IVP]}$$

We will see later how to deal with boundary conditions on an interval $0 \leq x \leq L$.

- <u>Steps</u>: 1) Study properties satisfied by the solutions.
  2) Solve a particular case.
  3) Use 1) and 2) to solve the general case.

2.4.1 Invariance properties of the heat equation

Let's see some properties that solutions of the heat equation,

$u_t - K u_{xx} = 0$ [HHE]

must satisfy.

1) Translations in space:

$u(x,t)$ solution to [HHE] $\Rightarrow$ So is the function $u(x-y,t)$ for any $y \in \mathbb{R}$.

Check:

$$\frac{\partial}{\partial t}(u(x-y,t)) - k\frac{\partial^2}{\partial x^2}(u(x-y,t)) = u_t(x-y,t) - k\,u_{xx}(x-y,t) = 0$$

since $u_t(x,t) - k u_{xx}(x,t) = 0$ for all $x \in \mathbb{R}$.

2) Differentiation:

$u(x,t)$ solution to [HHE] $\Rightarrow$ So are $u_x, u_t, u_{xx}, \dots$

3) Linearity:

If $u_1, u_2, \dots, u_n$ solutions of [HHE] $\Rightarrow$ So is

$$u = c_1 u_1 + c_2 u_2 + \dots + c_n u_n$$

for any $c_1, \dots, c_n \in \mathbb{R}$.

4) Scaling:

$u(x,t)$ solution of [HHE] $\Rightarrow$ So is $v(x,t) = u(\sqrt{a}x, at)$

for all $a > 0$.

<u>Check</u>:

$$\Rightarrow \left.\begin{aligned} v_t(x,t) &= \frac{\partial}{\partial t} u(\sqrt{a}x, at) = u_t(\sqrt{a}x, at)\cdot a \\ v_x(x,t) &= \frac{\partial}{\partial x} u(\sqrt{a}x, at) = u_x(\sqrt{a}x, at)\cdot\sqrt{a} \\ v_{xx}(x,t) &= u_{xx}(\sqrt{a}x, at)\sqrt{a}\sqrt{a} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow v_t(x,t) - k v_{xx}(x,t) = a\Big(\underbrace{u_t(\sqrt{a}x, at) - k u_{xx}(\sqrt{a}x, at)}_{=0}\Big) = 0$$

5) Convolution:

If $S(x,t)$ is a solution to [HHE] $\Rightarrow$ So is its "convolution" with any function $g(x)$ (as long the integral converges).

$\downarrow$

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\, g(y)\, dy \equiv \underbrace{(S*g)(x)}_{\text{Convolution of } S \text{ and } g.}$$

Check: (Assuming all integrals converge)

$$v_t(x,t) - k\, v_{xx}(x,t) = \int_{-\infty}^{\infty} S_t(x-y,t)\, g(y)\, dy - k \int_{-\infty}^{\infty} S_{xx}(x-y,t)\, g(y)\, dy =$$

$$= \int_{-\infty}^{\infty} \big(S_t(x-y,t) - k\, S_{xx}(x-y,t) \big)\, g(y)\, dy \underset{\uparrow}{=} 0 .$$

S satisfies [HHE].

Remark: We can understand this as a limiting case of the linearity property

$$\int_{-\infty}^{\infty} S(x-y,t)\, g(y)\, dy = \lim_{L\to\infty} \int_{-L}^{L} S(x-y,t)\, g(y)\, dy =$$

$$= \lim_{L\to\infty} \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{S(x-y_i)}_{\substack{\text{solution to the} \\ \text{heat eq. due to translation prop.}}} \underbrace{g(y_i)\,\Delta y}_{\text{coefficient}} ,$$

$$\left(\begin{array}{l} \text{where } -b = y_1 < y_2 < \dots < y_n = b \text{ is a partition of } [-b,b], \\ \Delta y = y_{i+1} - y_i \end{array} \right).$$

2.4.2 Solution to a particular case.

Consider as initial data the Heaviside function:

$$H(x) = \begin{cases} 1 & x > 0, \\ 0 & x < 0. \end{cases}$$

So we want to solve:

$$\left.\begin{array}{l} Q_t - KQ_{xx} = 0 \\ Q(x,0) = H(x) \end{array}\right\} \begin{pmatrix} -\infty < x < \infty \\ t > 0 \end{pmatrix} \quad [IVPQ]$$

where we are denoting the solution by Q.

Step 1: Using the previous properties to reduce the PDE to an ODE.

→ We want to show that the solution Q only depends on one quantity, in particular, that

$$Q(x,t) = g\left(\frac{x}{\sqrt{t}}\right)$$

This would allow us to reduce the PDE for Q to an ODE for g.

→ So let's show it:

If we have a solution $Q(x,t)$ to [JVPa], this means that:

- $Q(x,t)$ solves [HHE]
- $Q(x,0) = H(x)$

From the scaling property, $Q(\sqrt{a}\,x, at)$ also solves [HHE].

But, moreover,

this is why we choose the special case $\phi(x) = H(x)$.

$$Q(\sqrt{a}\,x, 0) = H(\sqrt{a}) \overset{\downarrow}{=} H(x),$$

so $Q(\sqrt{a}\,x, at)$ is another solution to [JVPa].

→ The uniqueness property then ensures that

$$Q(\sqrt{a}\,x, at) = Q(x,t) \quad \forall x \in \mathbb{R},\ t>0$$

That is, Q is invariant under the scaling $(x,t) \mapsto (\sqrt{a}\,x, at)$.

→ Let's see that this implies that Q can only depend on the quantity $\frac{x}{\sqrt{t}}$, i.e., that $Q(x,t) = g\left(\frac{x}{\sqrt{t}}\right)$, for some function $g(s)$ of one variable.

Indeed, define $g(s) = Q(s,1)$. For any point (x,t), choose $a = \frac{1}{t}$. Then,

$$Q(x,t) = Q\left(\frac{1}{\sqrt{t}}x, \frac{1}{t}t\right) = Q\left(\frac{x}{\sqrt{t}}, 1\right) = \mathfrak{g}\left(\frac{x}{\sqrt{t}}\right).$$

In summary, we are saying that if we know the function $\mathfrak{g}(s)$ for all $s \in \mathbb{R}$, we know the function $Q(x,t)$ for all $x \in \mathbb{R}$, $t > 0$.

→ Finally, let's find the ODE for $\mathfrak{g}$:

<u>Remark</u>: <u>Only</u> for simplicity in the computations, we will use instead the function

$$g(s) = \mathfrak{g}(\sqrt{4k}\, s).$$

That is,

$$g\left(\frac{x}{\sqrt{4kt}}\right) = \mathfrak{g}\left(\frac{x}{\sqrt{t}}\right) = Q(x,t)$$

• Then, use the chain rule: $\left(\text{Denote } s = \frac{x}{\sqrt{4kt}}\right)$

$$Q_t(x,t) = \frac{\partial}{\partial t}\left(g\left(\frac{x}{\sqrt{4kt}}\right)\right) = g'(s)\left(\frac{-1}{2}\,\frac{4kx}{(4kt)^{3/2}}\right) = \frac{-1}{2t}\,\frac{x}{\sqrt{4kt}}\,g'(s),$$

$$Q_x(x,t) = \frac{\partial}{\partial x}\left(g\left(\frac{x}{\sqrt{4kt}}\right)\right) = g'(s)\,\frac{1}{\sqrt{4kt}}$$

$$Q_{xx}(x,t) = g''(s)\,\frac{1}{4kt},$$

Therefore,

$$Q_t - kQ_{xx} = 0 = \frac{-1}{2t}\frac{x}{\sqrt{4kt}}g'(s) - \frac{k}{4kt}g''(s) = 0 \leadsto$$

$$\leadsto \frac{-s}{2t}g'(s) - \frac{1}{4t}g''(s) = 0 \leadsto$$

$$\leadsto \| g''(s) + 2s\,g'(s) = 0 \|$$

- Solution to the ODE:

Call $\omega(s) = g'(s) \leadsto \omega'(s) + 2s\,\omega(s) = 0 \leadsto$

$$\Rightarrow \frac{d\omega}{\omega} = -2s\,ds \Rightarrow \log(\omega) = -s^2 + c \leadsto \| \omega(s) = c_1 e^{-s^2}$$

That is,

$$g'(s) = c_1 e^{-s^2} \Rightarrow g(s) = c_1 \int e^{-s^2} ds + c_2$$

$\rightarrow$ Let's go back: recall that $Q(x,t) = g\left(\frac{x}{\sqrt{4kt}}\right)$, so

$$\boxed{Q(x,t) = c_1 \int_0^{x/\sqrt{4kt}} e^{-s^2} ds + c_2}$$

Now we need to impose the initial condition from [IVPQ].

• Initial condition: $H(x) = \begin{cases} 1 & x > 0, \\ 0 & x < 0. \end{cases}$

$$\left.\begin{aligned} 1 &= \lim_{t \to 0^+} Q(x,t) = c_1 \int_0^{+\infty} e^{-s^2}\,ds + c_2 \overset{(*)}{=} c_1 \frac{\sqrt{\pi}}{2} + c_2, \\ 0 &= \lim_{t \to 0^-} Q(x,t) = c_1 \int_0^{-\infty} e^{-s^2}\,ds + c_2 \overset{(*)}{=} -c_1 \frac{\sqrt{\pi}}{2} + c_2. \end{aligned}\right\} \Rightarrow$$

[*] Exercise: $\int_0^{\infty} e^{-s^2}\,ds = \frac{\sqrt{\pi}}{2}$.

$\Rightarrow c_1 = \frac{1}{\sqrt{\pi}}, \; c_2 = \frac{1}{2} \Rightarrow$

$$\boxed{Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-s^2}\,ds} \quad (t > 0)$$

Solution to [IVP 2].

• So now we can move to step 3): Finding the solution to [IVP], i.e., with general initial condition $\phi(x)$.

2.4.3) Solution to [IVP]

Let $Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-s^2} ds \quad (t>0)$.

Define: $S(x,t) = \frac{\partial}{\partial x}(Q(x,t)) = Q_x(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$.

Then, by the invariance properties,

→ $S(x,t)$ is also a solution to the heat equation.

→ So is $u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy \quad (t>0)$.

Moreover:

Claim: The solution to the general initial value problem for the heat equation, [IVP], is

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)dy} \quad (t>0)$$

Proof.

Since we already know this verifies the heat equation, we only need to check the initial condition.

Recall that: $u(x,t)=\int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy$, with

$S(x,t) = Q_x(x,t)$.

Then,

$$u(x,t)=\int_{-\infty}^{\infty} Q_x(x-y,t)\phi(y)dy = -\int_{-\infty}^{\infty} Q_y(x-y)\phi(y)\,dy =$$

$$= \int_{-\infty}^{\infty} Q(x-y,t)\phi'(y)\,dy - \underbrace{Q(x-y,t)\phi(y)\Big]_{y=-\infty}^{y=\infty}}_{=0}$$

Recall that Q at $t=0$ was $H(x)$:

$$u(x,0) = \int_{-\infty}^{\infty} Q(x-y,0)\phi'(y)dy = \int_{-\infty}^{\infty} H(x-y)\phi'(y)\,dy = \int_{-\infty}^{x} \phi'(y)dy =$$

$$= \phi(y)\Big]_{y=-\infty}^{y=x} = \phi(x) \,/\!/ .$$

<u>Remark</u>: The function $\boxed{S(x,t) = \frac{1}{\sqrt{4\pi k t}}\, e^{\frac{-x^2}{4kt}}}$ is

called the kernel or the fundamental solution of the heat equation.

[*] $\displaystyle\int_{-\infty}^{\infty} e^{-s^2}\, ds = \frac{\sqrt{\pi}}{2}$.

↳ Denote $\displaystyle I = \int_{-\infty}^{\infty} e^{-x^2}\, dx = \int_{-\infty}^{\infty} e^{-y^2}\, dy$.

Then,

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right) \underset{\text{(Fubini's theorem)}}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy$$

Change to polar coordinates : $r^2 = x^2 + y^2$ $\begin{pmatrix} x = r\cos\theta \\ y = r\sin\theta \end{pmatrix}$ →

$$I^2 = \int_0^{2\pi}\int_0^{+\infty} e^{-r^2}\, r\, dr\, d\theta = 2\pi\left[\frac{-1}{2} e^{-r^2}\right]_0^{+\infty} = 2\pi\left(0 + \frac{1}{2}\right) = \pi \to$$

$$\Rightarrow I = \sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2}\, dx \,//.$$

→ Notice that $\displaystyle\int_0^{+\infty} e^{-x^2}\, dx \underset{\left(e^{-x^2} \text{ is an even function}\right)}{=} \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$.