MATH 425
LECTURE 5

: The Wave Equation

1.2.4/ Laplace equation.

In a stationary situation, the solution doesn't depend on time.

We can imagine a river flowing in such a way that two pictures at different times look exactly the same.

This can be translated into saying that $\partial_t \equiv 0$.

Of course, this does not mean that the river is not moving, but only that that movement is independent of time.

The heat and wave equation become the Laplace equation in the stationary case:

$$\left.\begin{array}{ll} \text{HEAT EQ:} & u_t = c^2 \Delta u \\ \text{WAVE EQ:} & u_{tt} = c^2 \Delta u \end{array}\right\} \xrightarrow{\partial_t \equiv 0} \boxed{\Delta u = 0} \text{ LAPLACE EQUATION.}$$

- Solutions to $\Delta u = 0$ are called <u>harmonic functions</u>.

1.3 | Boundary and initial conditions.

In general we need to give initial conditions, that is, the value of our unknown at $t=0$ (also of its "velocity" if there are second order derivatives in time), and boundary conditions.

<u>Ex:</u> $u_{tt} - u_{xx} = 0 \qquad t>0,\ 0<x<L$

$$\left.\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}\right\} \quad 0 \le x \le L$$

$$u(0,t) = 0 = u(L,t) = 0 \qquad t \geq 0$$

(fixed ends of the string)

The boundary conditions are typically classified in three types:

(D) Dirichlet condition: u is given

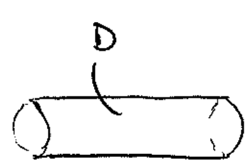
(N) Neumann condition: $n \cdot \nabla u$ is given (normal derivative, $\frac{\partial u}{\partial n}$)

(R) Robin condition: $n \cdot \nabla u + a(x,t)u$ is given

<u>Ex.</u> Heat equation

$$\left.\begin{aligned} u_t &= u_{xx} \quad \text{in } D,\ t>0 \\ u(x,0) &= f(x), \quad x \in D \\ u(x,t) &= 10, \quad x \in \partial D,\ t \geq 0 \end{aligned}\right\}$$

↳ temperature of the walls fixed at 10.



Neumann?
↳ insulation.

$\partial D \equiv$ boundary of the cylinder

1.4 | Well-posedness

A problem is well-posed in the Hadamard sense if:

1) Existence of solutions.

2) Uniqueness

3) Stability: small changes in the data turn into small changes in the solution.

- Clearly, given too few conditions (data) might turn into having many solutions, while given too much data might turn into no solution at all.

- The third condition is specially important to physics, as the (initial/boundary) conditions will come from experimental data, so one wants the solutions to be (more or less) the same regardless of measurement errors.

  ↳ Mathematically, one needs first to decide a definition of "small" (topology, norm, metric, ..)

- Example of ill-posedness: "antidiffusion"
  "Rayleigh-Taylor instabilities"
  ⋮

Chapter 2 : Wave and heat equation (introduction)

1.] Wave equation

We will consider (in this chapter) the wave equation on the real line. Although it is physically unreasonable in principle, two justifications:

1) Mathematically is simpler, and the fundamental properties appear already in this model.

2) We will see that what happens far from the boundary does not receive its influence (until enough time passes).

<u>Structure</u>: 1) Find general solution (on real line)

2) Solve the general IVP (initial value problem).

3) Causality → domain of dependence ("finite speed of propagation")

4) Conservation of energy ⇒ uniqueness.

1.1.1 General solution.

Wave equation: $u_{tt} - c^2 u_{xx} = 0 \quad -\infty < x < \infty$.

Solution: As we did in a previous homework, let's factorize the operator:

- $u_{tt} - c^2 u_{xx} = (\partial_t - c\partial_x)(\partial_t + c\partial_x) u = 0$.

Denote $v = u_t + c u_x$, then,

$\| \; v_t - c v_x = 0 \Rightarrow$ This is a simple transport equation.

Using the method of characteristics or the coordinate method, we find that

- $v(x,t) = f(x+ct)$

$\ulcorner \; v_t - c v_x = (1,-c)\cdot \nabla v = 0 \Rightarrow v$ is constant along the curves whose tangent vector is $(1,-c)$, i.e.,

$\frac{dx}{dt} = -c \Rightarrow x(t) = -ct + x_0 \Rightarrow v = f(x_0) = f(x+ct) \lrcorner$

- Going back, we have to solve now

$\| \; u_t + c u_x = f(x+ct) \Rightarrow$ Again a transport equation (now, inhomogeneous).

Let's solve this using the coordinate method:

$$\left.\begin{aligned}\tilde{t} &= t + cx \\ \tilde{x} &= ct - x\end{aligned}\right\} \Rightarrow \begin{aligned} x &= \frac{c\tilde{t} - \tilde{x}}{1+c^2} \\ t &= (\tilde{t} + c\tilde{x})/(1+c^2).\end{aligned}$$

$$\left.\begin{aligned} u_x &= \frac{\partial u}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial x} + \frac{\partial u}{\partial \tilde{t}}\frac{\partial \tilde{t}}{\partial x} = -u_{\tilde{x}} + c u_{\tilde{t}} \\ u_t &= \frac{\partial u}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial t} + \frac{\partial u}{\partial \tilde{t}}\frac{\partial \tilde{t}}{\partial t} = c u_{\tilde{x}} + u_{\tilde{t}} \end{aligned}\right\} \text{ thus,}$$

$u_t + c u_x = c u_{\tilde{x}} + u_{\tilde{t}} + c(-u_{\tilde{x}} + c u_{\tilde{t}}) = (1+c^2) u_{\tilde{t}}$, so the equation is rewritten as

$$(1+c^2) u_{\tilde{t}} = f\left(\frac{c\tilde{t} - \tilde{x} + c\tilde{t} + c^2\tilde{x}}{1+c^2}\right) \Rightarrow u_{\tilde{t}} = \frac{1}{1+c^2} f\left(\frac{2c\tilde{t} + (c^2-1)\tilde{x}}{1+c^2}\right)$$

Integrating in $\tilde{t}$,

$$u = \frac{1}{1+c^2}\int f\left(\frac{2c\tilde{t} + (c^2-1)\tilde{x}}{1+c^2}\right) d\tilde{t} = \frac{1}{2c} F\Bigg(\underbrace{\frac{2c\tilde{t} + (c^2-1)\tilde{x}}{1+c^2}}_{= x+ct}\Bigg) + \frac{g(\underbrace{\tilde{x}}_{\tilde{x} = ct - x})}{1+c^2},$$

with $F(s) = \int f(r)dr$.

We thus conclude that

$$\boxed{u(x,t) = f_1(x+ct) + f_2(ct - x)} \quad (f_1, f_2 \text{ arbitrary functions})$$

• **Remark:** Indeed, we could have seen this result using that the wave equation is linear and that

$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0$ holds if (1) $u_t + cu_x = 0$ or (2) $u_t - cu_x = 0$ $\Big\}$

(1) $\Rightarrow u_1 = f(x-ct)$ is a solution (the general ones for $(\partial_t + c\partial_x)u = 0$).

(2) $\Rightarrow u_2 = g(x+ct)$ " " " (" " " for $(\partial_t - c\partial_x)u=0$).

So all possible solutions are linear combinations of those two.

• **Remark:** We could have introduced the characteristics coordinates at the beginning,

$\left.\begin{array}{l} \xi = x + ct \\ \eta = x - ct \end{array}\right\}$ and solve the wave equation in these new variables.

Let's see:

$\left.\begin{array}{l} \partial_x = \partial_\xi + \partial_\eta \\ \partial_t = c\partial_\xi - c\partial_\eta \end{array}\right\}$ so,

$\partial_t - c\partial_x = \cancel{c\partial_\xi} - c\partial_\eta - \cancel{c\partial_\xi} - c\partial_\eta = -2c\,\partial_\eta$ (typo in book, p. 34)

$\partial_t + c\partial_x = c\partial_\xi - \cancel{c\partial_\eta} + c\partial_\xi + \cancel{c\partial_\eta} = 2c\,\partial_\xi$

Thus,

$u_{tt} - c^2 u_{xx} = (\partial_t - c\partial_x)(\partial_t + c\partial_x)u = -4c^2\partial_\xi\partial_\eta u = -4c^2 u_{\xi\eta} = 0 \Rightarrow$

$\Rightarrow u_{\xi\eta} = 0 \Rightarrow u_\xi = c(\xi) \Rightarrow u(\xi,\eta) = \int c(\xi)\,d\xi + g(\eta) = f(\xi) + g(\eta)$

(rename)

That is, $u(x,t) = \underbrace{f(x+ct)}_{\text{(... to the left ...)}} + \underbrace{g(x-ct)}_{\text{wave (of arbitrary shape) travelling to the right at speed } c}$ //.

1.2 | Initial Value Problem

$$\left.\begin{aligned} &u_{tt} = c^2 u_{xx} \qquad x \in \mathbb{R} \\ &u(x,0) = \phi(x) \\ &u_t(x,0) = \psi(x) \end{aligned}\right\}$$

<u>Solution</u>: $u(x,t) = f(x+ct) + g(x-ct)$, f, g?

- $u(x,0) = f(x) + g(x) = \phi(x)$
- $u_t(x,0)$?

$\hookrightarrow u_t(x,t) = f'_*(x+ct)\cdot c + g'_*(x-ct)(-c) \Rightarrow$

$\Rightarrow u_t(x,0) = c\, f'_*(x) - c\, g'_*(x) = \psi(x)$

That is,

$$\left.\begin{aligned} \phi(x) &= f(x) + g(x) \\ \psi(x) &= c f'(x) - c g'(x) \end{aligned}\right\}$$ Think of this as a 2×2 system, with $f(x), g(x)$ the unknowns.

Take derivatives in the first one:

$$\left.\begin{aligned}\phi'(x) &= f'(x) + g'(x) \\ \frac{1}{c}\psi(x) &= f'(x) - g'(x)\end{aligned}\right\} \Rightarrow \begin{aligned} 2f'(x) &= \phi'(x) + \frac{1}{c}\psi(x) \\ 2g'(x) &= \phi'(x) - \frac{1}{c}\psi(x)\end{aligned} \Rightarrow$$

(integration gives)

$$f(x) = \frac{1}{2}\phi(x) + \frac{1}{2c}\int_0^x \psi(s)\,ds + A,$$

$$g(x) = \frac{1}{2}\phi(x) - \frac{1}{2c}\int_0^x \psi(s)\,ds + B.$$

We had that $f(x) + g(x) = \phi(x)$, thus $A + B = 0$.

Therefore, we can conclude that

$$u(x,t) = f(x+ct) + g(x-ct) =$$

$$\left\| \; u(x,t) = \frac{1}{2}\Big(\phi(x+ct) + \phi(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds \; \right\|$$ D'Alembert solution (1746)

<u>Remark:</u> If $\phi \in C^2$, $\psi \in C^1$, one sees that u has continuous second derivatives in x, t.

Thus, D'Alembert solution is indeed a solution (in that case).

($\hookrightarrow$ Check that it is a solution from the eq.)

Example: Solve

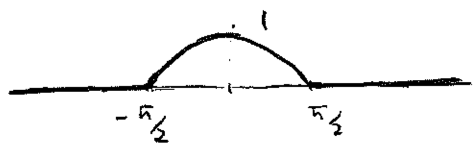
$$u_{tt} = c^2 u_{xx}$$

$$u(x,0) = \phi(x) = \begin{cases} \text{sen}\left(x + \frac{\pi}{2}\right), & |x| < \frac{\pi}{2} \\ 0, & |x| > 1 \end{cases}$$

$$\left(u_t(x,0) = \psi(x,0) = 0\right)$$

Solution:

Initial shape of the string:

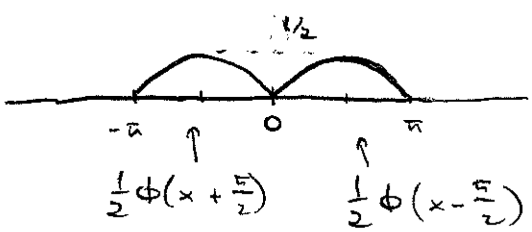


(**Remark**: ϕ is not differentiable, but everything "makes sense" anyway. We will see later in the course how to do it rigorously $\to$ weak solutions).

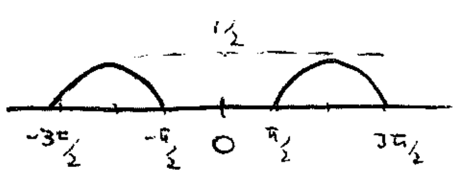
$$u(x,t) = \frac{1}{2}\Big(\phi(x+ct) + \phi(x-ct)\Big)$$

$\hookrightarrow$ The initial shape travels to the left and right with constant velocity c and half the initial amplitude.

$t = \frac{\pi}{2c}$:



$t = \frac{\pi}{c}$:



1.3 | Causality: domain of dependence / influence

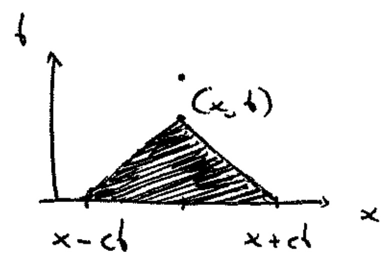
Q: Given a point (x,t), what part of the initial data affects the value $u(x,t)$?

Recall $u(x,t) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$.

- We see that the initial position of the string at points $x+ct$, $x-ct$ are needed to know u (the position) at (x,t).
- Also, we need to know the initial velocity at $(x-ct, x+ct)$.

Therefore, the closed interval $[x-ct, x+ct]$ are the points that determined $u(x,t)$.
This is called the domain of dependence of the point (x,t)

Sometimes domain of dependence is used to denote the entire region of the x-t plane, "the past history" of the point (x,t):



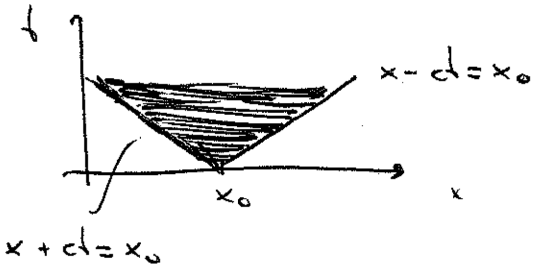
The question can also be asked inversely:

The initial data at x_0 affects which points x at later times t?

We already have the answer: the points (x,t) /

$x - ct \leq x_0 \leq x + ct$, that is,

$\| x_0 - ct \leq x \leq x_0 + ct \|$ region of influence of the point x_0



This is showing us again that the wave equation transfers the information at a finite speed.

So, for example, if the initial data is zero outside a region (say $|x| \geq R$), then $u(x,t) = 0$ for $|x| > R + ct$.

1.4] Energy conservation and uniqueness.

Let's define the total energy as

$$e(t) = \int_{-\infty}^{\infty} \frac{1}{2}\Big(\underbrace{u_t^2(x,t)}_{\text{Kinetic}} + \underbrace{c^2 u_x^2(x,t)}_{\text{"potential"}}\Big)\,dx$$

$$\left.\begin{array}{l} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{array}\right\}$$

<u>Conservation of energy</u>: If $e(0) = \frac{1}{2}\int_{-\infty}^{\infty}\left(\psi(x)^2 + c^2\phi'(x)^2\right)dx < \infty$,

then

$$e(t) = \frac{1}{2}\int_{-\infty}^{\infty}\left(u_t(x,t)^2 + c^2 u_x(x,t)^2\right)dx = e(0) < \infty \quad \text{for all } t > 0.$$

<u>Proof:</u>

We will assume that $\phi(x)$, $\psi(x)$ are bounded functions that vanish outside the interval $[a,b]$, just so that the integrals are convergent $\leadsto e(0)$ ✓.

Therefore we know that $u(x,t)$ also vanish outside $[a-ct, b+ct]$ $\leadsto$

$\leadsto e(t)$ ✓.

To show $e(t) = e(0)$, let's show that $\frac{de}{dt} \equiv 0$.

$$\frac{de}{dt} = \int_{-\infty}^{\infty}\left(u_t u_{tt} + c^2 u_x u_{xt}\right)dx = \int_{-\infty}^{\infty} u_t u_{tt}\,dx - \int_{-\infty}^{\infty} c^2 u_{xx} u_t\,dx + \cancel{u_t u_x}\Big]_{-\infty}^{\infty} =$$

$$= \int_{-\infty}^{\infty} u_t (u_{tt} - c^2 u_{xx})\, dx = 0 \text{//.}$$

- <u>Uniqueness</u>: There is only one finite-energy solution of the IVP

$$\left.\begin{aligned} &u_{tt} - c^2 u_{xx} = 0 \\ &u(x,0) = \phi(x) \\ &u_t(x,0) = \psi(x) \end{aligned}\right\}$$

Proof:

Suppose we have two $u_1(x,t), u_2(x,t)$.

Let $v = u_1 - u_2$, then

$$\left.\begin{aligned} &v_{tt} - c^2 v_{xx} = 0 \\ &v(x,0) = 0 = v_t(x,0) \end{aligned}\right\}$$

So $e_v(0) = 0 \Rightarrow e_v(t) = 0 \Rightarrow v_t \equiv 0 \equiv v_x \Rightarrow v \equiv 0$ //.