2 First-order linear equations

Last time we saw how some simple PDEs can be reduced to ODEs, and subsequently solved using ODE methods. For example, the equation

$$u_x = 0 \tag{2.1}$$

has “constant in x” as its general solution, and hence u depends only on y, thus $u(x, y) = f(y)$ is the general solution, with f an arbitrary function of a single variable. Today we will see that any linear first order PDE can be reduced to an ordinary differential equation, which will then allow as to tackle it with already familiar methods from ODEs.

Let us start with a simple example. Consider the following constant coefficient PDE

$$au_x + bu_y = 0. \tag{2.2}$$

Here a and b are constants, such that $a^2+b^2 \neq 0$, i.e. at least one of the coefficients is nonzero (otherwise this would not be a differential equation). Using the inner (scalar or dot) product in $\mathbb{R}^2$, we can rewrite the left hand side of (2.2) as

$$(a, b) \cdot (u_x, u_y) = 0, \qquad \text{or} \qquad (a, b) \cdot \nabla u = 0.$$

Denoting the vector $(a, b) = \mathbf{v}$, we see that the left hand side of the above equation is exactly $D_{\mathbf{v}}u(x, y)$, the directional derivative of u in the direction of the vector $\mathbf{v}$. Thus the solution to (2.2) must be constant in the direction of the vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$.

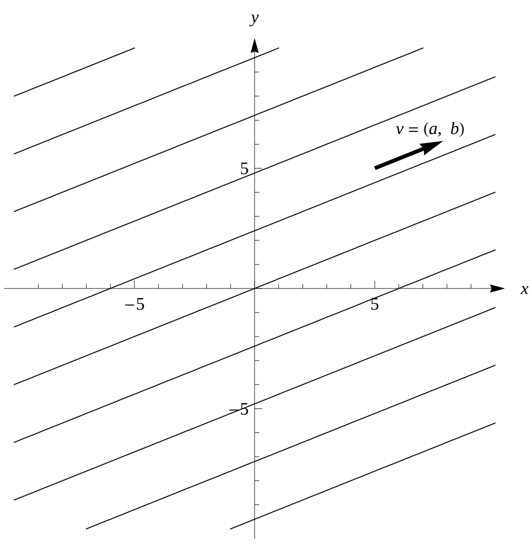

Figure 2.1: Characteristic lines $bx - ay = c$.

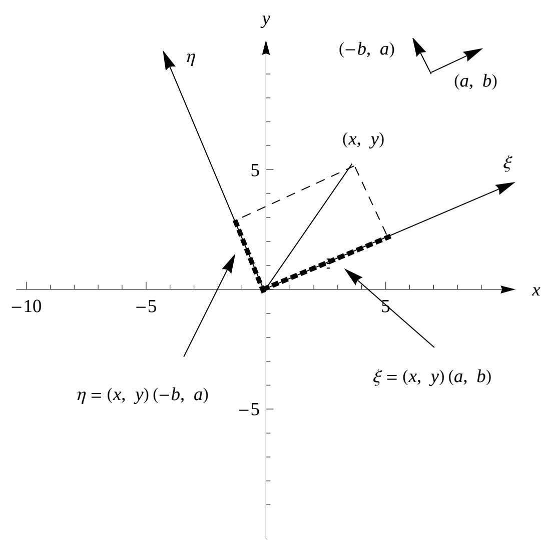

Figure 2.2: Change of coordinates.

The lines parallel to the vector $\mathbf{v}$ have the equation

$$bx - ay = c, \tag{2.3}$$

since the vector $(b, -a)$ is orthogonal to $\mathbf{v}$, and as such is a normal vector to the lines parallel to $\mathbf{v}$. In equation (2.3) c is an arbitrary constant, which uniquely determines the particular line in this family of parallel lines, called *characteristic lines* for the equation (2.2).

As we saw above, $u(x, y)$ is constant in the direction of $\mathbf{v}$, hence also along the lines (2.3). The line containing the point (x, y) is determined by $c = bx - ay$, thus u will depend only on $bx - ay$, that is

$$u(x, y) = f(bx - ay), \tag{2.4}$$

where f is an arbitrary function. One can then check that this is the correct solution by plugging it into the equation. Indeed,

$$a\partial_x f(bx - ay) + b\partial_y f(bx - ay) = abf'(bx - ay) - baf'(bx - ay) = 0.$$

The geometric viewpoint that we used to arrive at the solution is akin to solving equation (2.1) simply by recognizing that a function with a vanishing derivative must be constant. However one can approach equation (2.2) from another perspective, by trying to reduce it to an ODE.

2.1 The method of characteristics

To have an ODE, we need to eliminate one of the partial derivatives in the equation. But we know that the directional derivative vanishes in the direction of the vector (a, b). Let us then make a change of the coordinate system to one that has its "x-axis" parallel to this vector, as in Figure 2. In this coordinate system

$$(\xi, \eta) = ((x, y) \cdot (a, b), (x, y) \cdot (b, -a)) = (ax + by, bx - ay).$$

So the change of coordinates is

$$\begin{cases} \xi = ax + by, \\ \eta = bx - ay. \end{cases} \tag{2.5}$$

To rewrite the equation (2.2) in this coordinates, notice that

$$u_x = u_\xi \frac{\partial \xi}{\partial x} + u_\eta \frac{\partial \eta}{\partial x} = au_\xi + bu_\eta,$$
$$u_y = u_\xi \frac{\partial \xi}{\partial y} + u_\eta \frac{\partial \eta}{\partial y} = bu_\xi - au_\eta.$$

Thus,

$$0 = au_x + bu_y = a(au_\xi + bu_\eta) + b(bu_\xi - au_\eta) = (a^2 + b^2)u_\xi.$$

Now, since $a^2 + b^2 \neq 0$, then, as we anticipated,

$$u_\xi = 0,$$

which is an ODE. We can solve this last equation just as we did in the case of equation (2.1), arriving at the solution

$$u(\xi, \eta) = f(\eta).$$

Changing back to the original coordinates gives $u(x, y) = f(bx - ay)$. This is the same solution that we derived with the geometric deduction. This method of reducing the PDE to an ODE is called *the method of characteristics*, and the coordinates (ξ, η) given by formulas (2.5) are called *characteristic coordinates*.

Example 2.1. Find the solution of the equation $3u_x - 5u_y = 0$ satisfying the condition $u(0, y) = \sin y$.

From the above discussion we know that u will depend only on $\eta = -5x - 3y$, so $u(x, y) = f(-5x - 3y)$. The solution also has to satisfy the additional condition (called initial condition), which we verify by plugging in $x = 0$ into the general solution.

$$\sin y = u(0, y) = f(-3y).$$

So $f(z) = \sin(-\frac{z}{3})$, and hence $u(x, y) = \sin\left(\frac{5x+3y}{3}\right)$, which one can verify by substituting into the equation and the initial condition. □

2.2 General constant coefficient equations

We can easily adapt the method of characteristics to general constant coefficient linear first-order equations

$$au_x + bu_y + cu = g(x, y). \tag{2.6}$$

Recall that to find the general solution of this equation it is enough to find the general solution of the homogeneous equation

$$au_x + bu_y + cu = 0, \tag{2.7}$$

and add to this a particular solution of the inhomogeneous equation (2.6). Notice that in the characteristic coordinates (2.5), equation (2.7) will take the form

$$(a^2 + b^2)u_\xi + cu = 0, \qquad \text{or} \qquad u_\xi + \frac{c}{a^2 + b^2}u = 0,$$

which can be treated as an ODE in ξ. The solution to this ODE has the form

$$u_h(\xi, \eta) = e^{-\frac{c}{a^2+b^2}\xi} f(\eta),$$

with f again being an arbitrary single-variable function. Changing the coordinates back to the original (x, y), we will obtain the general solution to the homogeneous equation

$$u_h(x, y) = e^{-\frac{c(ax+by)}{a^2+b^2}} f(bx - ay).$$

You should verify that this indeed solves equation (2.7).

To find a particular solution of (2.6), we can use the characteristic coordinates to reduce it to the inhomogeneous ODE

$$(a^2 + b^2)u_\xi + cu = g(\xi, \eta), \qquad \text{or} \qquad u_\xi + \frac{c}{a^2 + b^2}u = \frac{g(\xi, \eta)}{a^2 + b^2}.$$

Having found the solution to the homogeneous ODE, we can find the solution to this inhomogeneous equation by e.g. variation of parameters. So the particular solution will be

$$u_p = e^{-\frac{c}{a^2+b^2}\xi} \int \frac{g(\xi, \eta)}{a^2 + b^2} e^{\frac{c}{a^2+b^2}\xi}\, d\xi.$$

The general solution of (2.6) is then

$$u(\xi, \eta) = u_h + u_p = e^{-\frac{c}{a^2+b^2}\xi} \left(f(\eta) + \int \frac{g(\xi, \eta)}{a^2 + b^2} e^{\frac{c}{a^2+b^2}\xi}\, d\xi \right).$$

To find the solution in terms of (x, y), one needs to first carry out the integration in ξ in the above formula, then replace ξ and η by their expressions in terms of x and y.

Example 2.2. Find the general solution of $-2u_x + 4u_y + 5u = e^{x+3y}$.

The characteristic change of coordinates for this equation is given by

$$\begin{cases} \xi = -2x + 4y, \\ \eta = 4x + 2y. \end{cases}$$

From these we can also find the expressions of x and y in terms of (ξ, η). In particular notice that $x + 3y = \frac{\xi+\eta}{2}$. In the characteristic coordinates the equation reduces to

$$20u_\xi + 5u = e^{\frac{\xi+\eta}{2}}.$$

The general solution of the homogeneous equation associated with the above equation is

$$u_h = e^{-\frac{1}{4}\xi} f(\eta),$$

and the particular solution will be

$$u_p = e^{-\frac{1}{4}\xi} \int \frac{e^{\frac{\xi+\eta}{2}}}{20} e^{\frac{1}{4}\xi}\, d\xi = e^{-\frac{1}{4}\xi} \cdot \frac{1}{15} e^{\frac{\eta}{2}} e^{\frac{3}{4}\xi} = e^{-\frac{1}{4}\xi} \cdot \frac{1}{15} e^{\frac{1}{4}(3\xi+2\eta)}.$$

Adding these two will give the general solution to the inhomogeneous equation

$$u(\xi,\eta) = e^{-\frac{1}{4}\xi} \left(f(\eta) + \frac{1}{15} e^{\frac{1}{4}(3\xi+2\eta)} \right).$$

Finally, substituting the expressions for ξ and η in terms of (x,y), we will obtain the solution

$$u(x,y) = e^{-\frac{1}{4}(2x-4y)} \left(f(4x+2y) + \frac{1}{15} e^{\frac{1}{4}(2x+16y)} \right).$$

You should check that this indeed solves the differential equation. □

2.3 Variable coefficient equations

The method of characteristics can be generalized to variable coefficient first-order linear PDEs as well, albeit the change of variables may no longer be orthogonal. The general variable coefficient linear first-order equations is

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y). \tag{2.8}$$

Let us first consider the following simple particular case

$$u_x + yu_y = 0. \tag{2.9}$$

Using our geometric intuition from the constant coefficient equations, we see that the directional derivative of u in the direction of the vector $\mathbf{v} = (1, y)$ is constant. Notice that the vector $\mathbf{v}$ itself is no longer constant, and varies with y. The curves that have $\mathbf{v}$ as their tangent vector have slope $\frac{y}{1}$, and thus satisfy

$$\frac{dy}{dx} = \frac{y}{1}.$$

We can solve this equation as an ODE, and obtain the general solution

$$y = Ce^x, \qquad \text{or} \qquad e^{-x}y = C. \tag{2.10}$$

As in the case of the constant coefficients, the solution to the equation (2.9) will be constant along these curves, called *characteristic curves*. This family of non-intersecting curves fills the entire coordinate plane, and the curve containing the point (x,y) is uniquely determined by $C = e^{-x}y$, which implies that the general solution to (2.9) is

$$u(x,y) = f(C) = f(e^{-x}y).$$

As always, we can check this by substitution.

$$u_x + yu_y = -f'(e^{-x}y)e^{-x}y + yf'(e^{-x}y)e^{-x} = 0.$$

Let us now try to generalize the method of characteristics to the equation

$$a(x,y)u_x + b(x,y)u_y = 0. \tag{2.11}$$

The idea is again to introduce new coordinates (ξ, η), which will reduce (2.11) to an ODE. Suppose

$$\begin{cases} \xi = \xi(x, y), \\ \eta = \eta(x, y) \end{cases} \tag{2.12}$$

gives such a change of variables. To rewrite the equation in this coordinates, we compute

$$u_x = u_\xi \xi_x + u_\eta \eta_x,$$
$$u_y = u_\xi \xi_y + u_\eta \eta_y,$$

and substitute these into equation (2.11) to get

$$(a\xi_x + b\xi_y)u_\xi + (a\eta_x + b\eta_y)u_\eta = 0.$$

Since we would like this to give us an ODE, say in ξ, we require the coefficient of u_η to be zero,

$$a\eta_x + b\eta_y = 0.$$

Without loss of generality, we may assume that $a \neq 0$ (locally). Notice that for curves $y(x)$ that have the slope $\frac{dy}{dx} = \frac{b}{a}$ we have

$$\frac{d}{dx}\eta(x, y(x)) = \eta_x + \eta_y \frac{dy}{dx} = \eta_x + \frac{b}{a}\eta_y = 0.$$

So the characteristic curves, just as before, are given by

$$\frac{dy}{dx} = \frac{b}{a}. \tag{2.13}$$

The general solution to this ODE will be $\eta(x, y) = C$, with $\eta_y \neq 0$ (otherwise $\eta_x = 0$ as well, and this will not be a solution). This is how we find the new variable η, for which our PDE reduces to an ODE. We choose $\xi(x, y) = x$ as the other variable. For this change of coordinates the Jacobian determinant is

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \eta_y \neq 0.$$

Thus, (2.12) constitutes a non-degenerate change of coordinates. In the new variables equation (2.11) reduces to

$$a(\xi, \eta)u_\xi = 0, \quad \text{hence} \quad u_\xi = 0,$$

which has the solution

$$u = f(\eta).$$

The general variable coefficient equation (2.8) in these coordinates will reduce to

$$a(\xi, \eta)u_\xi + c(\xi, \eta)u = d(\xi, \eta),$$

which is called the *canonical form* of equation (2.8). This equation, as in previous cases, can be solved by standard ODE methods.

Example 2.3. Find the general solution of the equation

$$xu_x - yu_y + y^2 u = y^2, \qquad x, y \neq 0.$$

Condition (2.13) in this case is $\frac{dy}{dx} = -\frac{y}{x}$. This is a separable ODE, which can be solved to obtain the general solution $xy = C$. Thus, our change of coordinates will be

$$\begin{cases} \xi = x, \\ \eta = xy. \end{cases}$$

In these coordinates the equation takes the form

$$\xi u_\xi + \frac{\eta^2}{\xi^2} u = \frac{\eta^2}{\xi^2}, \quad \text{or} \quad u_\xi + \frac{\eta^2}{\xi^3} u = \frac{\eta^2}{\xi^3}.$$

Using the integrating factor

$$e^{\int \frac{\eta^2}{\xi^3}\, d\xi} = e^{-\frac{\eta^2}{2\xi^2}},$$

the above equation can be written as

$$\left(e^{-\frac{\eta^2}{2\xi^2}} u \right)_\xi = e^{-\frac{\eta^2}{2\xi^2}} \frac{\eta^2}{\xi^3}.$$

Integrating both sides in ξ, we arrive at

$$e^{-\frac{\eta^2}{2\xi^2}} u = \int e^{-\frac{\eta^2}{2\xi^2}} \frac{\eta^2}{\xi^3}\, d\xi = e^{-\frac{\eta^2}{2\xi^2}} + f(\eta).$$

Thus, the general solution will be given by

$$u(\xi, \eta) = e^{\frac{\eta^2}{2\xi^2}} \left(f(\eta) + e^{-\frac{\eta^2}{2\xi^2}} \right) = e^{\frac{\eta^2}{2\xi^2}} f(\eta) + 1.$$

Finally, substituting the expressions of ξ and η in terms of (x, y) into the solution, we obtain

$$u(x, y) = f(xy) e^{\frac{y^2}{2}} + 1.$$

One should again check by substitution that this is indeed a solution to the PDE. □

2.4 Conclusion

The method of characteristics is a powerful method that allows one to reduce *any* first-order linear PDE to an ODE, which can be subsequently solved using ODE techniques. We will see in later lectures that a subclass of second order PDEs – second order hyperbolic equations can be also treated with a similar characteristic method.

3 Method of characteristics revisited

3.1 Transport equation

A particular example of a first order constant coefficient linear equation is the transport, or advection equation $u_t + cu_x = 0$, which describes motions with constant speed. One way to derive the transport equation is to consider the dynamics of the concentration of a pollutant in a stream of water flowing through a thin tube at a constant speed c.

Let $u(t, x)$ denote the concentration of the pollutant in gr/cm (unit mass per unit length) at time t. The amount of pollutant in the interval $[a, b]$ at time t is then

$$\int_a^b u(x, t)\, dx.$$

Due to conservation of mass, the above quantity must be equal to the amount of the pollutant after some time h. After the time h, the pollutant would have flown to the interval $[a + ch, b + ch]$, thus the conservation of mass gives

$$\int_a^b u(x, t)\, dx = \int_{a+ch}^{b+ch} u(x, t + h)\, dx.$$

To derive the dynamics of the concentration $u(x, t)$, differentiate the above identity with respect to b to get

$$u(b, t) = u(b + ch, t + h).$$

Notice that this equation asserts that the concentration at the point b at time t is equal to the concentration at the point $b + ch$ at time $t + h$, which is to be expected, due to the fact that the water containing the pollutant particles flows with a constant speed. Since b is arbitrary in the last equation, we replace it with x. Now differentiate both sides of the equation with respect to h, and set h equal to zero to obtain the following differential equation for $u(x, t)$.

$$0 = cu_x(x, t) + u_t(x, t),$$

or

$$u_t + cu_x = 0. \tag{3.1}$$

Since equation (3.1) is a first order linear PDE with constant coefficients, we can solve it by the method of characteristics. First, we rewrite the equation as

$$(1, c) \cdot \nabla u = 0,$$

which implies that the slope of the characteristic lines is given by

$$\frac{dx}{dt} = \frac{c}{1}.$$

Integrating this equation, one arrives at the equation for the characteristic lines

$$x = ct + x(0), \tag{3.2}$$

where $x(0)$ is the coordinate of the point at which the characteristic line intersects the x-axis. The solution to the PDE (3.1) can then be written as

$$u(t, x) = f(x - ct) \tag{3.3}$$

for any arbitrary single-variable function f.

Let us now consider a particular initial condition for $u(t,x)$

$$u(0,x) = \begin{cases} x & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

According to (3.3), $u(0,x) = f(x)$, which determines the function f. Having found the function from the initial condition, we can now evaluate the solution $u(t,x)$ of the transport equation from (3.3). Indeed

$$u(t,x) = f(x-ct) = \begin{cases} x-ct & 0 < x-ct < 1 \\ 0 & \text{otherwise} \end{cases}$$

Noticing that the inequalities $0 < x - ct < 1$ imply that x is in-between ct and $ct+1$, we can rewrite the above solution as

$$u(t,x) = \begin{cases} x-ct & ct < x < ct+1, \\ 0 & \text{otherwise,} \end{cases}$$

which is exactly the initial function $u(0,x)$, given by (3.4), moved to the right along the x-axis by ct units. Thus, the *initial data* $u(0,x)$ travels from left to right with constant speed c.

We can alternatively understand the dynamics by looking at the characteristic lines in the xt coordinate plane. From (3.2), we can rewrite the characteristics as

$$t = \frac{1}{c}(x - x(0)).$$

Along these characteristics the solution remains constant, and one can obtain the value of the solution at any point (t,x) by tracing it back to the x-axis:

$$u(t,x) = u(t-t, x-ct) = u(0, x(0)).$$

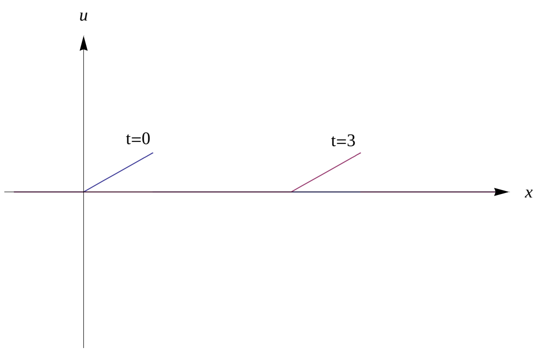

Figure 3.1: $u(t,x)$ at two different times t.

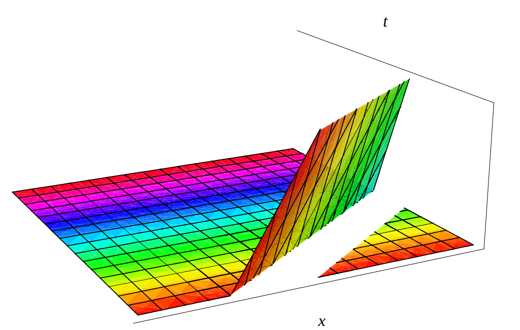

Figure 3.2: The graph of $u(t,x)$ colored with respect to time t.

Figure 1 gives the graphs of $u(t,x)$ at two different times, while Figure 3.1 gives the three dimensional graph of $u(t,x)$ as a function of two variables.

3.2 Quasilinear equations

We next look at a simple nonlinear equation, for which the method of characteristics can be applied as well. The general first order quasilinear equation has the following form

$$a(x,y,u)u_x + b(x,y,u)u_y = g(x,y,u).$$

We can see that the highest order derivatives, in this case the first order derivatives, enter the equation linearly, although the coefficients depend on the unknown u. A very particular example of first order quasilinear equations is the inviscid Burger's equation

$$u_t + uu_x = 0. \tag{3.5}$$

As before, we can rewrite this equation in the form of a dot product, which is a vanishing condition for a certain directional derivative

$$(1, u) \cdot (u_t, u_x) = 0, \qquad \text{or} \qquad (1, u) \cdot \nabla u = 0.$$

This shows that the tangent vector of the characteristic curves, $\mathbf{v} = (1, u)$, depends on the unknown function u.

3.3 Rarefaction

Let as now look at a particular initial condition, and try to construct solutions along the characteristic curves. Suppose

$$u(0, x) = \begin{cases} 1 & \text{if } x < 0, \\ 2 & \text{if } x > 0. \end{cases} \tag{3.6}$$

The slope of the characteristic curves satisfies

$$\frac{dx}{dt} = u(t, x(t)) = u(0, x(0)).$$

Here we used the fact that the directional derivative of the solution vanishes in the direction of the tangent vector of the characteristic curves. This implies that the solution remains constant along the characteristics, i.e. $u(t, x(t))$ remains constant for all values of t. We can find the equation of the characteristics by integrating the above equation, which gives

$$x(t) = u(0, x(0))t + x(0). \tag{3.7}$$

Using the initial condition (3.6), this equation will become

$$x(t) = \begin{cases} t + x(0) & \text{if } x(0) < 0, \\ 2t + x(0) & \text{if } x(0) > 0. \end{cases}$$

Thus, the characteristics have different slopes depending on whether they intersect the x axis at a positive, or negative intercept $x(0)$. We can express the characteristic lines to give t as a function of x, so that the initial condition is defined along the horizontal x axis.

$$t = \begin{cases} x - x(0) & \text{if } x(0) < 0, \\ \frac{1}{2}(x - x(0)) & \text{if } x(0) > 0. \end{cases} \tag{3.8}$$

Some of the characteristic lines corresponding to the initial condition (3.6) are sketched in Figure 3 below. The solid lines are the two families of characteristics with different slopes.

Notice that in this case the waves originating at $x(0) > 0$ move to the right faster than the waves originating at points $x(0) < 0$. Thus an increasing gap is formed between the faster moving wave front and the slower one. One can also see from Figure 3, that there are no characteristic lines from either of the two families given by (3.8) passing through the origin, since there is a jump discontinuity at $x = 0$ in the initial condition (3.6). In fact, in this case we can imagine that there are infinitely many characteristics originating from the origin with slopes ranging between $\frac{1}{2}$ and 1 (the dotted lines in Figure 3). The proper way to see this is to notice that in the case of $x(0) = 0$, (3.7) implies that

$$u = \frac{x}{t}, \qquad \text{if} \qquad t < x < 2t.$$

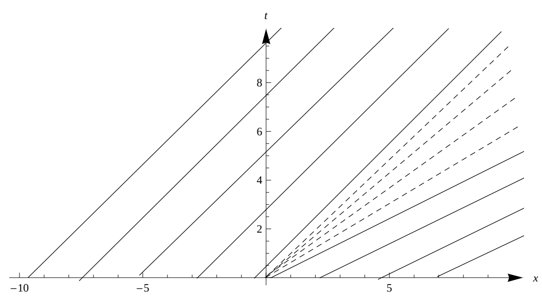

Figure 3.3: Characteristic lines illustrating rarefaction.

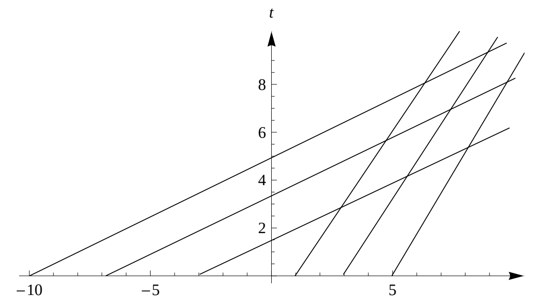

Figure 3.4: Characteristic lines illustrating shock wave formation.

This type of waves, which arise from decompression, or *rarefaction* of the medium due to the increasing gap formed between the wave fronts traveling at different speeds are called *rarefaction waves*. Putting all the pieces together, we can write the solution to equation (3.5) satisfying initial condition (3.6) as follows.

$$u(t,x) = \begin{cases} 1 & \text{if } x < t, \\ \frac{x}{t} & \text{if } t < x < 2t, \\ 2 & \text{if } x > 2t. \end{cases}$$

3.4 Shock waves

A completely opposite phenomenon to rarefaction is seen when one has a faster wave moving from left to right catching up to a slower wave. To illustrate this, let us consider the following initial condition for the Burger's inviscid equation

$$u(0,x) = \begin{cases} 2 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases} \tag{3.9}$$

Then the characteristic lines (3.7) will take the form

$$x(t) = \begin{cases} 2t + x(0) & \text{if } x(0) < 0, \\ t + x(0) & \text{if } x(0) > 0. \end{cases}$$

Or expressing t in terms of x, we can write the equations as

$$t = \begin{cases} \frac{1}{2}(x - x(0)) & \text{if } x(0) < 0, \\ x - x(0) & \text{if } x(0) > 0. \end{cases} \tag{3.10}$$

Thus, the characteristics origination from $x(0) < 0$ have smaller slope (corresponding to faster speed), than the characteristics originating from $x(0) > 0$. In this case the characteristics from the two families will intersect eventually, as seen in Figure 4. At the intersection points the solution u becomes multivalued, since the point can be traced back along either of the characteristics to an initial value of either 1, or 2, given by the initial condition (3.9). This phenomenon is known as *shock waves*, since the faster moving wave catches up to the slower moving wave to form a multivalued (multicrest) wave.

3.5 Conclusion

We saw that the method of characteristics can be generalized to quasilinear equations as well. Using the method of characteristics for the inviscid Burger's equations, we discovered that in the case of nonlinear equations one may encounter characteristics that diverge from each other to give rise to an unexpected solution in the widening region in-between, as well as intersecting characteristics, leading to multivalued solutions. These are nonlinear phenomena, and do not arise in the study of linear PDEs.