MATH 425 LECTURE 2

: The Method of Characteristics.

Last day: ODEs

- First order
  - separable
  - linear: $u'(x) + p(x)u(x) = g(x)$

  $\hookrightarrow u(x) = e^{-\int p(x)}\left(\int e^{\int p(x)} g(x)\,dx + c\right)$

- Second order: linear and cte coeff. $\to u'' + bu' + cu = 0$

  (three cases $\rightsquigarrow$ roots of $r^2 + br + c = 0$)

- Examples (uniqueness)

1) $\left\{\begin{array}{l} u'' + u = 0 \\ u(0) = 0 \\ u(\pi) = 0 \end{array}\right.$

Sol: $r^2 + 1 = 0 \Rightarrow r = \pm i$

$\hookrightarrow$ $\left.\begin{array}{l} u(x) = c_1 \cos x + c_2 \sin x \\ u(0) = c_1 = 0 \\ u(\pi) = -c_1 = 0 \end{array}\right\} \Rightarrow u(x) = c_2 \sin x,$

$c_2 \in \mathbb{R}$ arbitrary.

(u is <u>not unique</u>).

2) Existence and uniqueness

$\left.\begin{array}{l} u'' - cu = 0 \\ u(a) = p \\ u(b) = q \end{array}\right\}$ with $c \geq 0$, $a \neq b$

Existence: find one solution $\rightarrow r^2 - c = 0 \rightsquigarrow r = \pm c \Rightarrow$
$(c \neq 0)$
$$\Rightarrow u(x) = c_1 e^{cx} + c_2 e^{-cx}$$

$$\left.\begin{array}{l} u(a) = p \Rightarrow p = c_1 e^{ca} + c_2 e^{-ca} \\ u(b) = g \rightarrow g = c_1 e^{cb} + c_2 e^{-cb} \end{array}\right\} \Rightarrow \underbrace{\begin{bmatrix} e^{ac} & e^{-ac} \\ e^{bc} & e^{-bc} \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} p \\ g \end{bmatrix}$$

$\det(A) \neq 0$ (for $c \neq 0$).

If $c = 0$,

$$u'' = 0 \Rightarrow u = c_1 x + c_2$$

$$\left.\begin{array}{l} u(a) = p \Rightarrow p = c_1 a + c_2 \\ u(b) = g \rightarrow g = c_1 b + c_2 \end{array}\right\} \Rightarrow \begin{bmatrix} a & 1 \\ b & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} p \\ g \end{bmatrix}$$

Uniqueness:

Let $u_1 \neq u_2$ be two sol, and $v = u_1 - u_2$. Then $\left.\begin{array}{l} v'' - cv = 0 \\ v(a) = v(b) = 0 \end{array}\right\}$

1) Multiply by v: $v''v - cv^2 = 0$,

2) Integrate from a to b: $\int_a^b v''(x)v(x)\,dx = \int_a^b c\,v(x)^2\,dx$,

3) Integrate by parts:

$$0 \leq \int_a^b c\,v(x)^2\,dx = \int_a^b v''(x)v(x)\,dx = -\int_a^b (v'(x))^2\,dx + \Big[v'(x)\cancelto{0}{v(x)}\Big]_a^b = -\int_a^b v'(x)^2\,dx \leq 0$$

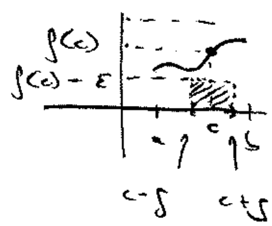
$$\Rightarrow \int_a^b v(x)^2\,dx = 0 \overset{(*)}{\Rightarrow} \cancel{\Leftarrow}\ v \equiv 0.$$

(*) Is that always true?

$$\left.\begin{array}{l}\displaystyle\int_a^b f(x)\,dx = 0\\[2ex] f(x)\geq 0 \quad \forall x\in[a,b]\end{array}\right\} \Rightarrow f(x)=0 \quad \forall x\in[a,b] \qquad ?$$

→ "<u>Yes</u>": if f is continuous.

<u>Proof</u>: Let's proceed by contradiction.



Assume $\exists c\in(a,b) \,/\, f(c)>0$.

Let $\varepsilon = \frac{f(c)}{2}$. Then,

$\exists \delta>0 \,/\, |x-c|<\delta \Rightarrow |f(x)-f(c)|<\varepsilon$.

That is, for $x\in(c-\delta, c+\delta)$, $f(x)\in(f(c)-\varepsilon, f(c)+\varepsilon)$.

In particular, $f(x) > f(c)-\varepsilon = \frac{f(c)}{2}$.

Therefore,

$$\int_a^b f(x)\,dx \geq \int_{c-\delta}^{c+\delta} f(x)\,dx > \int_{c-\delta}^{c+\delta} \frac{f(c)}{2}\,dx = \delta f(c) > 0 \ \rightarrow\!\leftarrow .$$

Contradiction.

So $f(x)=0 \quad \forall x\in(a,b)$.

By continuity, one also has $f(a)=0=f(b)$.

$\boxed{\text{Chapter 1}}$: Introduction to PDEs.

We can think of a PDE as an ODE with several variables. That is, given a function $u(x, y, \dots)$, a PDE is an equation relating the variables $x, y, \dots$, the function u and its partial derivatives up to order n. We call <u>order</u> of the PDE to this n,

$$F(x, y, u(x,y), u_x(x,y), u_y(x,y)) = 0$$] PDE in two variables of order one.

How to solve PDEs?

Let's start with some ~~(...)~~ simple examples:

<u>Ex:</u> 1) $u_{yy}(x,y) = 0$

<u>Sol:</u> $u_y(x,y) = f(x) \rightsquigarrow u(x,y) = f(x)y + g(x)$

2) $u_{xx} + u = 0$

$\hookrightarrow u(x,y) = c_1(y)\cos x + c_2(y)\sin x$

3) $u_{xy} + 5u_y = 1$

Two ways $\rightarrow$

3.1) Integrate in y: $u_x + 5u = y + f(x)$

and solve in x: $u(x,y) = e^{-5x}\left(\int e^{5x}(y + f(x))\,dx + g(y)\right) =$

$$= e^{-5x}\int e^{5x} y\,dx + F(x) + e^{-5x} g(y) =$$

$$= \frac{y}{5} + F(x) + e^{-5x} g(y).$$

3.2) Denote $v = u_y$,

$$v_x + 5v = 1 \Rightarrow v = e^{-5x}\left(\int e^{5x}\,dx + c(y)\right) = \frac{1}{5} + e^{-5x} c(y) \Rightarrow$$

$$\Rightarrow u(x,y) = \frac{y}{5} + F(x) + e^{-5x} g(y) \,//.$$

4) a) Show there exists a unique solution to

$$\left.\begin{array}{l} u_x = 3x^2 y + y \\ u_y = x^3 + x \end{array}\right\} \text{ with } u(0,0) = 0.$$

b) Prove that $\left.\begin{array}{l} u_x = 2.9x^2 y + y \\ u_y = x^3 + x \end{array}\right\}$ can't have a solution.

Sol: a) $u(x,y) = \cancel{x^3 y + y\int f(x) + c(y)}\; x^3 y + xy + c(y)$

$\hookrightarrow u_y = x^3 + x + c'(y) = x^3 + x \Rightarrow c'(y) = 0 \Rightarrow c(y) = c.$

Since $u(0,0) = 0 \Rightarrow c = 0 \Rightarrow u(x,y) = yx^3 + xy$

b) $u_{xy} = 2.9x^2 + 1$

$u_{yx} = 3x^2 + 1$ $\Big\} \Rightarrow u_{xy} \neq u_{yx}$!

1.1) Linear first-order pdes.

They take the form $a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y)$

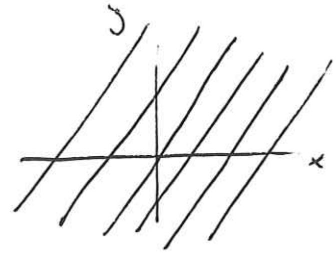
→ If $f \equiv 0$, "homogeneous" equation.

1.1.1) The constant coefficient case.

Ex: $3u_x + 4u_y = 0$

Notice that $(3,4)\cdot\nabla u = 0$ → i.e., u is constant in the direction $(3,4)$. That is, on lines where $(3,4)$ is tangent:

$$\frac{dy}{dx} = \frac{4}{3} \Rightarrow y = \frac{4x}{3} + c$$



The value of u along a line is constant. So u only depends on which line we are → That's given by $c = y - \frac{4x}{3}$ →

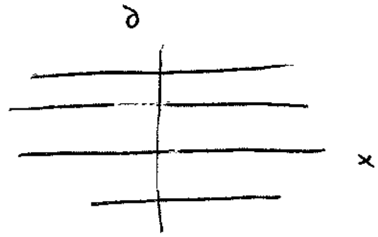
$$u(x,y) = f\left(y - \frac{4x}{3}\right), \text{ for any function } f.$$

• The coordinate method (or change of variable).

Compare the previous example with the simplest case,

$u_x = 0.$

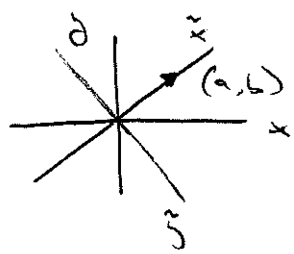
Here, $u(x,y) = f(y)$.



We can write the equation $a u_x + b u_y = 0$ in a new coordinate system:

$a u_x + b u_y = 0 \rightsquigarrow (a,b)\cdot\nabla u = 0 \rightsquigarrow$ ~~tangent~~ derivative in (a,b) direction is zero. Choose that line as the new axis:

$$\begin{cases} \tilde{x} = ax + by \\ \tilde{y} = bx - ay \end{cases}$$



In these new variables, we can check that $u_{\tilde{x}} = 0$:

$$\begin{cases} u_x = \dfrac{\partial u}{\partial \tilde{x}}\dfrac{\partial \tilde{x}}{\partial x} + \dfrac{\partial u}{\partial \tilde{y}}\dfrac{\partial \tilde{y}}{\partial x} = u_{\tilde{x}}\, a + b u_{\tilde{y}} \\[2ex] u_y = \dfrac{\partial u}{\partial \tilde{x}}\dfrac{\partial \tilde{x}}{\partial y} + \dfrac{\partial u}{\partial \tilde{y}}\dfrac{\partial \tilde{y}}{\partial y} = b u_{\tilde{x}} - a u_{\tilde{y}} \end{cases}$$

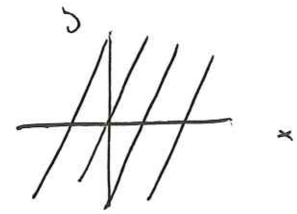
$\Rightarrow a u_x + b u_y = a^2 u_{\tilde{x}} + \cancel{ab} u_{\tilde{y}} + b^2 u_{\tilde{x}} - \cancel{ab} u_{\tilde{y}} = (a^2 + b^2) u_{\tilde{x}} = 0$

So $u_{\tilde{x}} = 0 \Rightarrow$ $u = f(\tilde{y}) = f(bx - ay)$.

Ex: Solve $2u_x + 4u_y = 0$ with $u(x,0) = \cos(x)$.

Sol: u is cte along the line spanned by $(2,4)$, i.e.,

line: $\frac{dy}{dx} = \frac{4}{2} \Rightarrow y = 2x + c$



So $u(x,y) = f(c) = f(y-2x)$

Now,
$$u(x,0) = f(-2x) = \cos(x) \Rightarrow f(x) = \cos\left(\frac{-x}{2}\right)$$

Thus, $u(x,y) = \cos\left(\frac{2x-y}{2}\right)$ $\|$.

Ex: Solve $u_x + u_y + u = e^{x+2y}$ with $u(x,0) = 0$.

Sol: We will use the coordinate method.

$$\left.\begin{aligned} \tilde{x} &= x + y \\ \tilde{y} &= x - y \end{aligned}\right\} \rightarrow \left.\begin{aligned} x &= \tfrac{1}{2}\tilde{x} + \tfrac{1}{2}\tilde{y} \\ y &= \tfrac{1}{2}\tilde{x} - \tfrac{1}{2}\tilde{y} \end{aligned}\right]$$

Now,

$$\left.\begin{aligned} u_x &= \frac{\partial u}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial x} + \frac{\partial u}{\partial \tilde{y}}\frac{\partial \tilde{y}}{\partial x} = u_{\tilde{x}} + u_{\tilde{y}} \\ u_y &= \frac{\partial u}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial y} + \frac{\partial u}{\partial \tilde{y}}\frac{\partial \tilde{y}}{\partial y} = u_{\tilde{x}} - u_{\tilde{y}} \end{aligned}\right\} \quad u_x + u_y = 2\tilde{u}_x$$

So we obtain, $2u_{\tilde{x}} + u = e^{\frac{3}{2}\tilde{x} - \frac{1}{2}\tilde{y}} \Rightarrow$

$$\Rightarrow u_{\tilde{x}} + \frac{1}{2}u = \frac{1}{2}e^{\frac{3}{2}\tilde{x}} e^{-\tilde{y}/2} \Rightarrow$$

$$\Rightarrow u = e^{-\frac{1}{2}\tilde{x}} \int e^{\frac{1}{2}\tilde{x}} \frac{1}{2} e^{\frac{3}{2}\tilde{x}} e^{-\tilde{y}/2} \, d\tilde{x} =$$

$$= e^{-\frac{1}{2}\tilde{x}} e^{-\frac{\tilde{y}}{2}} \frac{1}{2}\frac{1}{2} e^{2\tilde{x}} + \frac{1}{2} e^{-\tilde{x}/2} c(\tilde{y}) =$$

$$= \frac{1}{4} e^{-\tilde{y}/2} e^{\frac{3}{2}\tilde{x}} + \frac{1}{2} e^{-\tilde{x}/2} c(\tilde{y})$$

Thus,

$$u(x,y) = \frac{1}{4} e^{-\frac{x}{2} + \frac{y}{2}} e^{\frac{3}{2}x + \frac{3}{2}y} + \frac{1}{2} e^{-\frac{x}{2} - \frac{y}{2}} c(x-y)$$

$$\cdot\ u(x,0) = 0 = \frac{1}{4} e^{x} + \frac{1}{2} e^{-\frac{x}{2}} c(x) \Rightarrow c(x) = -\frac{1}{2} e^{x} e^{x/2} = -\frac{e^{3/2x}}{2}.$$

Finally,

$$u(x,y) = \frac{1}{4} e^{x+2y} \cancel{- \frac{1}{4} e^{x-\frac{y}{2}}} - \frac{1}{4} e^{-\frac{x}{2} - \frac{y}{2}} e^{\frac{3}{2}(x-y)} =$$

$$= \frac{1}{4} e^{x+2y} - \frac{1}{4} e^{x-2y} \quad //.$$

__Check__

__Q__: Solve it using method of characteristics.

1.1.2 Variable coefficient case.

Here $a(x,y)$, $b(x,y)$, $c(x,y)$ need not be constants.

We need to use the method of characteristic (not valid the coordinate method anymore).

First case: $a(x,y)u_x + b(x,y)u_y = 0$.

We can use the same idea: $(a(x,y), b(x,y)) \cdot \nabla u = 0 \to$

$\to u$ is constant along the "lines" defined by their tangent vector $(a(x,y), b(x,y))$.

That is, the characteristic curves are ~~g~~ defined by

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)} \quad \text{ODE (first order)} \rightsquigarrow y(x).$$

And then, $u(x, y(x)) = \text{cte}$.

Example: $u_x - y u_y = 0$

$$\frac{dy}{dx} = -y \Rightarrow \frac{dy}{y} = -dx \to \log y = -x + c \Rightarrow y = c e^{-x}$$

So, $u(x, ce^{-x}) = \text{cte} = u(0, c) \Rightarrow u(x,y) = u(0, e^{x} y) \Rightarrow$

$$\Rightarrow u(x,y) = f(e^{x} y)$$

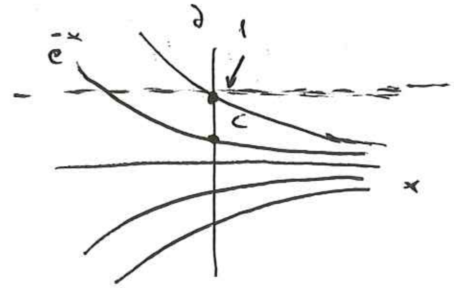
Question: If $u(x,1) = 2e^{x}$, in which region of the xy plane the solution is uniquely defined?

$u(x,y) = f(e^{x}y)$

$u(x,1) = f(e^{x}) = 2e^{x}$

$\hookrightarrow f(s) = 2s$

$\Downarrow$

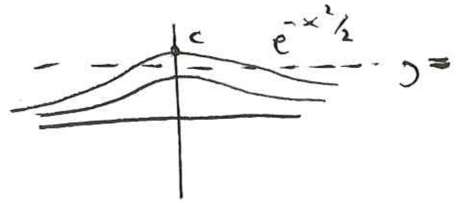


$\boxed{u(x,y) = 2e^{x}y}$ uniquely defined for the region $\{(x,y) \in \mathbb{R}^2 : y > 0\}$

Ex: $\frac{1}{x}u_x - yu_y = 0$, a) $u(x,1) = e^{x^2}$ b) $u(1,y) = e^{y^2}$

$\frac{dy}{dx} = \frac{-y}{1/x} = -yx \rightarrow \frac{dy}{y} = -x\,dx \rightarrow \ln y = -\frac{x^2}{2} + c \rightarrow y = ce^{-x^2/2}$

$u(x,y) = f(ye^{x^2/2})$



a) $u(x,1) = f(e^{x^2/2}) = e^{x^2}$

$\hookrightarrow f(s) = s^2 \rightarrow u(x,y) = y^2e^{x^2}$ uniquely determined in the region $y \geq e^{-x^2/2}$

b) $u(1,y) = e^{y^2} = f(ye^{1/2}) \rightarrow f(s) = e^{s^2e^{-1}}$

$\hookrightarrow u(x,y) = e^{e^{-1}y^2e^{x^2}} = e^{y^2e^{x^2-1}}$ uniquely defined in all $y \geq 0$.