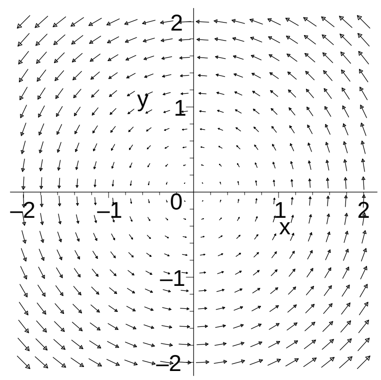
The dual nature of vector fields

A *vector field* is a function that assigns to each point in a domain a vector based at that point. For example,

$$\mathbf{V} = y\mathbf{i} - x\mathbf{j}$$

is a vector field defined on the xy-plane. To visualize a vector field, imagine drawing the vector that corresponds to each point based at the point. So the vector field $\mathbf{V}$ above could be graphed as follows:



From the picture, it becomes clear that a vector field can be used to represent a "velocity field" – one imagines the steady motion of a fluid or a gas, and the vector field at a point gives the velocity of the particle that happens to be at that point at any given time. The adjective "steady" refers to the fact that the vector field does not depend on the time. Thinking of a vector field in this manner immediately brings out the question of finding the trajectories of the individual particles (atoms, molecules,...) in the fluid. We want to find the curves that "follow the arrows", in other words, we would like a family of paramatrized curves $x = f(t)$, $y = g(t)$, so that the velocity vector of these curves exactly match up with the vector field at each point through which the curves pass.

If an arbitrary vector field $\mathbf{V}$ is given in components as

$$\mathbf{V} = v_1(x, y)\mathbf{i} + v_2(x, y)\mathbf{j},$$

the the problem of finding these "integral curves" of the vector field (or some would say the "flow" determined by the vector field, and still others would say the "orbits" of the vector field), means to solve the system of differential equations

$$\frac{dx}{dt} = v_1(x, y) \qquad \frac{dy}{dt} = v_2(x, y).$$

If we want the integral curve that passes through a specific point, say (a, b) at time

$t = 0$, we can also provide the initial conditions

$$x(0) = a, \qquad y(0) = b.$$

This is precisely enough information to specify an integral curve uniquely.

For example, for the vector field $\mathbf{V} = y\mathbf{i} - x\mathbf{j}$ pictured above, it looks as though the integral curves should be circles centered at the origin, and traversed clockwise. We can verify this by solving the initial-value problems

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x, \qquad x(0) = a, \quad y(0) = 0$$

for various values of a. To solve the system of differential equations, substitute dx/dt for y in the second equation (since this is information provided by the first equation) to get the second-order problem

$$\frac{d^2x}{dt^2} = -x,$$

which has the general solution $x(t) = c_1 \cos t + c_2 \sin t$. In order to have $x(0) = a$, we will need $c_1 = a$, and in order to have $y(0) = x'(0) = 0$, we will need $c_2 = 0$. The solution of the initial-value problem is thus

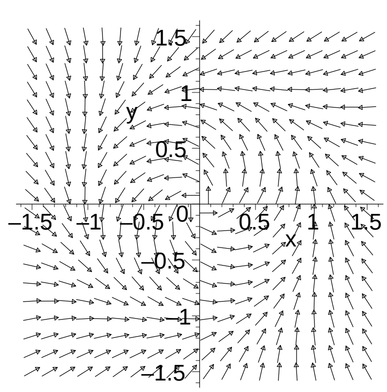
$$x(t) = a \cos t, \qquad y(t) = -a \sin t,$$

which is the circle of radius a centered at the origin and traversed clockwise, as we had anticipated. Notice there is a "singular orbit", namely if we choose $a = 0$, then the particle just sits at the origin. In fact, we can view the whole "dynamical system" represented by the vector field as a constant angular speed rotation in the clockwise direction around the origin.

A second example is the vector field

$$\mathbf{V} = (x - y - x^3 - xy^2)\mathbf{i} + (x + y - x^2y - y^3)\mathbf{j},$$

pictured here (although in this picture we have drawn all the arrows the same length so that you can see the shape better):



This vector field is more mysterious, although it looks as though the flow is generally counterclockwise, out from the origin and in from infinity, converging perhaps on the unit circle.

To see this more clearly, we'll resort to using the standard polar coordinates r and θ, so that $x = r\cos\theta$ and $y = r\sin\theta$. Ordinarily, to find the integral curves of the vector field, we would have to solve the system:

$$\begin{aligned}\frac{dx}{dt} &= x - y - x^3 - xy^2 \\ \frac{dy}{dt} &= x + y - x^2y - y^3\end{aligned}$$

but this will simplify a great deal in polar coordinates. To change from x, y to r, θ, notice that we will have:

$$\begin{aligned}\frac{dx}{dt} &= \frac{d(r\cos\theta)}{dt} = \cos\theta\,\frac{dr}{dt} - r\sin\theta\,\frac{d\theta}{dt} \\ \frac{dy}{dt} &= \frac{d(r\sin\theta)}{dt} = \sin\theta\,\frac{dr}{dt} + r\cos\theta\,\frac{d\theta}{dt}.\end{aligned}$$

Multiply the first equation by $\cos\theta$ and the second by $\sin\theta$ and add, and you will have solved for dr/dt. Likewise, multiply the first equation by $-\sin\theta/r$ and the second by $\cos\theta/r$ and add and you will have solved for $d\theta/dt$, as follows:

$$\begin{aligned}\frac{dr}{dt} &= \cos\theta\,\frac{dx}{dt} + \sin\theta\,\frac{dy}{dt} \\ \frac{d\theta}{dt} &= -\frac{\sin\theta}{r}\frac{dx}{dt} + \frac{\cos\theta}{r}\frac{dy}{dt}.\end{aligned}$$

And since $x - y - x^3 - xy^2 = r(\cos\theta - \sin\theta) - r^3\cos\theta$, and $x + y - x^2y - y^3 = r(\cos\theta + \sin\theta) - r^3\sin\theta$, we calculate that our original ODE system is equivalent to

$$\begin{aligned}\frac{dr}{dt} &= r - r^3 \\ \frac{d\theta}{dt} &= 1.\end{aligned}$$

The second equation gives us the general counter-clockwise "swirl" effect. And since $r - r^3$ is positive for $r < 1$, and negative for $r > 1$ we see that trajectories inside the unit circle increase their radius and so push out toward the unit circle, and trajectories outside the unit circle have decreasing radius and so move inward toward the unit circle. The unit circle is called a "limit cycle" of this flow.

The title of this section refers to the "dual nature" of vector fields. The second major role played by vector fields is that they provide directions along which to differentiate functions of several variables. Recall from multivariable calculus that the derivative of the function $u(x, y)$ in the direction of the vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ is

$$\nabla_{\mathbf{v}} u = \mathbf{v} \cdot \nabla u = v_1 \frac{\partial u}{\partial x} + v_2 \frac{\partial u}{\partial y},$$

where ∇u is the gradient of u. So if we have a vector *field* $\mathbf{v}$, and a (scalar-valued) function u, the directional derivative $\nabla_{\mathbf{v}} u$ is a new scalar-valued function.

There are two important ways to interpret this. First (this idea is usually emphasized in beginning graduate courses in differential geometry and topology), vector fields act as *differential operators* on functions. We will consider this point of view in the next section as we begin to study first-order partial differential equations. Second, we can understand the directional derivative aspect of $\nabla_{\mathbf{v}} u$ by considering the behavior of u along integral curves of the vector field $\mathbf{v}$.

In particular, if $(x(t), y(t))$ is an integral curve of $\mathbf{v}$ with $(x(0), y(0)) = (a, b)$, then the restriction of u to the integral curve is obtained by replacing x by $x(t)$ and y by $y(t)$ in the definition of u. So we'll write

$$u(t) = u(x(t), y(t)).$$

Using the chain rule for partial derivatives, we see that

$$\begin{aligned}
\frac{du(t)}{dt} &= \frac{d(u(x(t), y(t))}{dt} \\
&= \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \\
&= v_1 \frac{\partial u}{\partial x} + v_2 \frac{\partial u}{\partial y} \\
&= \mathbf{v} \cdot \nabla u = \nabla_{\mathbf{v}} u.
\end{aligned}$$

To go from the second line to the third line, we used the fact that $(x(t), y(t))$ is an integral curve of $\mathbf{v}$. So we say that $\nabla_{\mathbf{v}} u$ is the derivative of u in the direction of $\mathbf{v}$ to mean that $\nabla_{\mathbf{v}} u$ is the derivative of u along the integral curves of $\mathbf{v}$.

First-order partial differential equations

We can use what we know about vector fields to study linear first-order partial differential equations. Such an equation has the form:

$$a(x, y)\frac{\partial u}{\partial x} + b(x, y)\frac{\partial u}{\partial y} + c(x, y)u = f(x, y).$$

The left side of this equation is clearly linear in u, and so if $f(x, y) = 0$, solutions of this (*homogeneous* linear) equation will have the superposition property: linear combinations of solutions (with constant coefficients) will again be solutions.

To get a problem with a unique solution, we need to specify initial values for u – these will be values specified along some curve in the xy-plane. This is the analog of the initial data at a point that we specified for an ODE. If u were a function of n variables (instead of just 2), we would have to specify a function of $n-1$ variables (along some hypersurface in n-dimensional space) to get a problem with a unique solution.

There is a new complication with the initial data for a PDE, though. We cannot specify the initial data along any arbitrary curve. As we will see in more detail, we must avoid having the initial curve be tangent to the integral curves of the vector field $\mathbf{v} = a(x, y)\mathbf{i} + b(x, y)\mathbf{j}$. In the context of first-order PDEs, these integral curves are called the *characteristics* of the PDE.

It's easiest to see all this in an example. Let's start simple, and try and solve the PDE:

$$3u_x + 4u_y = 0,$$

where we have resorted to using subscript notation for partial derivatives, so u_x stands for $\partial u/\partial x$ and so forth. The expression on the left side of the PDE is the derivative of u in the direction of the vector

$$\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}.$$

Since the integral curves of $\mathbf{v}$ are lines with slope $4/3$, we will say that these lines are the *characteristic lines* of the PDE. The PDE says that u is constant along these lines.

Now, the simplest (non-constant) function that is constant along the characteristic lines is $4x - 3y$, since if one starts with given values of x and y, and then increases y by $4t$ and x by $3t$ (so as to go along the line with slope $4/3$), the value of $4x - 3y$ will remain unchanged. So $u = 4x - 3y$ is one solution of $3u_x + 4u_y = 0$. More generally, for *any* function $f(q)$ of a single variable, we will have that $u(x, y) = f(4x - 3y)$ is a solution of the PDE (use the chain rule for partial derivatives to check that this is true).

This expression, $u(x, y) = f(4x - 3y)$, is the *general solution* of our PDE - it contains in it one arbitrary function of one variable. This is a basic principle in PDEs - just as the general solution of a first-order ODE contains one arbitrary constant, the general solution of a first-order PDE contains one arbitrary function of one fewer variable than the unknown depends on. In fact, if you view a constant as a function of zero variables, then the ODE fact is a special case of the PDE fact.

Since we have the freedom to choose a function of one variable in the solution of our PDE, we expect that we can prescribe "initial data" along a curve in the

xy-plane. This is true, as long as the curve has the property that it intersects each characteristic curve of the PDE once, and is never tangent to the characteristic curves. For instance, the x-axis has this property with respect to the characteristic curves of $3u_x + 4u_y = 0$, so we can prescribe initial data along it, for instance $u(x, 0) = \cos(3x)$. To get the specific solution of the PDE that satisfies this additional condition, we simply substitute the condition into the general solution. It gives:

$$\cos(3x) = f(4x),$$

and so (substituting q for $4x$), $f(q) = \cos(3q/4)$. Therefore, the solution of the initial-value problem

$$3u_x + 4u_y = 0, \qquad u(x, 0) = \cos(3x)$$

is

$$u(x, y) = \cos(3(4x - 3y)/4) = \cos(3x - 9y/4).$$

Again, you can check that this solution satisfies the PDE and the initial condition.

To make things more interesting, let's consider solving the initial-value problem:

$$3u_x + 4u_y + 2u = x, \qquad u(x, 0) = e^{-x}.$$

For this equation, we'll use the general technique of *integrating along the characteristics*. To begin, we paramatrize the characteristics, using the variable t as the parameter. And we will do it in such a way that, along each characteristic, the point of intersection of the characteristic with the curve along which the initial data is prescribed occurs when $t = 0$. In particular, since the characteristics satisfy:

$$\frac{dx}{dt} = 3, \quad \frac{dy}{dt} = 4, \quad y(0) = 0,$$

we can write $x = 3t+C$, $y = 4t$. Then y will be zero when $t = 0$, and the characteristic will intersect the x-axis at the point $(C, 0)$.

Along a characteristic, the chain rule for partial derivatives tells us that

$$\frac{du}{dt} = 3u_x + 4u_y,$$

and so along each characteristic we need to solve the ODE

$$\frac{du}{dt} + 2u = 3t + C,$$

with initial data $u(0) = e^{-C}$. This is a linear equation, with integrating factor e^{2t}. The solution of this initial-value problem is

$$u(t) = \frac{C}{2} - \frac{3}{4} + \frac{3t}{2} + e^{-2t}(e^{-C} - \frac{C}{2} + \frac{3}{4}).$$

This is in fact the solution of the PDE, once we translate the t's and C's into x's and y's. Since

$$x = 3t + C, \qquad y = 4t$$

we have

$$t = \frac{y}{4}, \qquad C = x - \frac{3y}{4}.$$

And so we get the solution to the PDE problem:

$$\begin{aligned} u(x,y) &= \frac{1}{2}(x - \frac{3y}{4}) - \frac{3}{4} + \frac{3y}{8} + e^{-y/2}(e^{3y/4-x} - \frac{1}{2}(x - \frac{3y}{4}) + \frac{3}{4}) \\ &= \frac{x}{2} - \frac{3}{4} + e^{-y/2}(e^{3y/4-x} - \frac{x}{2} + \frac{3y}{8} + \frac{3}{4}). \end{aligned}$$

One further example, in which the vector field that defines the characteristics has variable coefficients:

$$2xu_x + 4u_y = 4x^2y^3u, \qquad u(x,0) = \cos x$$

This is still a linear equation, and exhibits all the complications that a linear equation can have. The problem is chosen somewhat carefully, because there are two things that could go wrong. First, one of the integrals (the ones that define the characteristic curves, or else the ones involved in solving the resulting first-order ODE for u) might be impossible to do in closed form. Second, the change of coordinates from x, y to the parameters used for the characteristic curves might be impossible to invert (so we would have the solution expressed in the parametric variables but would be unable to get back to x's and y's). But this example exhibits neither of those obstacles.

So we begin as usual by finding the characteristic curves, and we paramatrize them (with the variable t) in such a way that $t = 0$ corresponds to the point where the characteristic curve intersects the curve ($y = 0$, the x-axis) along which the initial data is given. The vector field that defines the characteristics is $\mathbf{v} = 2x\mathbf{i} + 4\mathbf{j}$, and so the characteristic curves are obtained by solving

$$\frac{dx}{dt} = 2x, \quad \frac{dy}{dt} = 4 \qquad x(0) = C, \quad y(0) = 0.$$

The solutions are $x = Ce^{2t}$, $y = 4t$. Therefore, along a characteristic, the differential equation becomes

$$\frac{du}{dt} = 4(C^2e^{4t})(64t^3), \qquad u(0) = \cos C$$

The differential equation is separable, and we rewrite it as

$$\frac{du}{u} = 256C^2t^3\,dt$$

Several integrations by parts later, we arrive at the general solution

$$\ln u = 256C^2(\frac{1}{4}t^3 - \frac{3}{16}t^2 + \frac{3}{32}t - \frac{3}{128})e^{4t} + \ln K.$$

And using $u(0) = \cos C$, we get $K = e^{6C^2} \cos C$, therefore

$$u(t, C) = (\cos C) \exp(6C^2 + Ce^{4t}(64s^3 - 48s^2 + 24s - 6)).$$

This is the solution, expressed in the t, C coordinate system. To get back to x, y, we note that

$$t = \frac{y}{4}, \quad C = xe^{-y/2}.$$

We substitute for t and C in the solution, and simplify a little, obtaining:

$$u(x, y) = \cos(xe^{-y/2}) \exp(6x^2e^{-y} + x^2y^3 - 3x^2y^2 + 6x^2y - 6x).$$