15 Heat with a source

So far we considered homogeneous wave and heat equations and the associated initial value problems on the whole line, as well as the boundary value problems on the half-line and the finite line (for wave only). The next step is to extend our study to the inhomogeneous problems, where an external heat source, in the case of heat conduction in a rod, or an external force, in the case of vibrations of a string, are also accounted for. We first consider the inhomogeneous heat equation on the whole line,

$$\begin{cases} u_t - ku_{xx} = f(x,t), & -\infty < x < \infty, t > 0, \\ u(x,0) = \phi(x), \end{cases} \tag{15.1}$$

where $f(x,t)$ and $\phi(x)$ are arbitrary given functions. The right hand side of the equation, $f(x,t)$ is called the *source* term, and measures the physical effect of an external heat source. It has units of heat flux (left hand side of the equation has the units of u_t, i.e. change in temperature per unit time), thus it gives the instantaneous temperature change due to an external heat source.

From the superposition principle, we know that the solution of the inhomogeneous equation can be written as the sum of the solution of the homogeneous equation, and a particular solution of the inhomogeneous equation. We can thus break problem (15.1) into the following two problems

$$\begin{cases} u^h_t - ku^h_{xx} = 0, \\ u^h(x,0) = \phi(x), \end{cases} \tag{15.2}$$

and

$$\begin{cases} u^p_t - ku^p_{xx} = f(x,t), \\ u^p(x,0) = 0. \end{cases} \tag{15.3}$$

Obviously, $u = u^h + u^p$ will solve the original problem (15.1).

Notice that we solve for the general solution of the homogeneous equation with arbitrary initial data in (15.2), while in the second problem (15.3) we solve for a particular solution of the inhomogeneous equation, namely the solution with zero initial data. This reduction of the original problem to two simpler problems (homogeneous, and inhomogeneous with zero data) using the superposition principle is a standard practice in the theory of linear PDEs.

We have solved problem (15.2) before, and arrived at the solution

$$u^h(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy, \tag{15.4}$$

where $S(x,t)$ is the heat kernel. Notice that the physical meaning of expression (15.4) is that the heat kernel averages out the initial temperature distribution along the entire rod.

Since $f(x,t)$ plays the role of an external heat source, it is clear that this heat contribution must be averaged out, too. But in this case one needs to average not only over the entire rod, but over time as well, since the heat contribution at an earlier time will effect the temperatures at all later times. We claim that the solution to (15.3) is given by

$$u^p(x,t) = \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)\,dy\,ds. \tag{15.5}$$

Notice that the time integration is only over the interval $[0,t]$, since the heat contribution at later times can not effect the temperature at time t. Combining (15.4) and (15.5) we obtain the following solution to the IVP (15.1)

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)\,dy\,ds, \tag{15.6}$$

or, substituting the expression of the heat kernel,

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y)\,dy + \int_0^t \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/4k(t-s)}}{\sqrt{4\pi k(t-s)}} f(y,s)\,dy\,ds.$$

One can draw parallels between formula (15.6) and the solution to the inhomogeneous ODE analogous to the heat equation. Indeed, consider the IVP for the following ODE.

$$\begin{cases} \frac{d}{dt}u(t) - Au(t) = f(t), \\ u(0) = \phi, \end{cases} \tag{15.7}$$

where A is a constant (more generally, for vector valued u, the equation will be a system of ODEs for the components of u, and A will be a matrix with constant entries). Using an integrating factor e^{-At}, the ODE in (15.7) yields

$$\frac{d}{dt}\left(e^{-At}u\right) = e^{-At}\frac{du}{dt} - Ae^{-At}u = e^{-At}(u' - Au) = e^{-At}f(t).$$

But then

$$e^{-At}u = \int_0^t e^{-As} f(s)\,ds + e^{-A\cdot 0}u(0),$$

and multiplying both sides by e^{At} gives

$$u(t) = e^{At}\phi + \int_0^t e^{A(t-s)} f(s)\,ds. \tag{15.8}$$

The operator $\mathscr{S}(t)\phi = e^{At}\phi$, called the *propagator* operator, maps the initial value ϕ to the solution of the homogeneous equation at later times. In terms of this operator, we can rewrite solution (15.8) as

$$u(t) = \mathscr{S}(t)\phi + \int_0^t \mathscr{S}(t-s) f(s)\,ds. \tag{15.9}$$

In the case of the heat equation, the heat propagator operator is

$$\mathscr{S}(t)\phi = \int_{-\infty}^{\infty} S(x-y,t)\phi(t)\,dy,$$

which again maps the initial data ϕ to the solution of the homogeneous equation at later times. Using the heat propagator, we can rewrite formula (15.6) in exactly the same form as (15.9).

We now show that (15.6) indeed solves the problem (15.1) by a direct substitution. Since we have solved the homogeneous equation before, it suffices to show that u^p given by (15.5) solves problem (15.3). Differentiating (15.5) with respect to t gives

$$\partial_t u^p = \int_{-\infty}^{\infty} S(x-y,0) f(y,t)\,dy + \int_0^t \int_{-\infty}^{\infty} \frac{\partial}{\partial t} S(x-y,t-s) f(y,s)\,dy\,ds.$$

Recall that the heat kernel solves the heat equation and has the Dirac delta function as its initial data, i.e. $S_t = kS_{xx}$, and $S(x-y,0) = \delta(x-y)$. Hence,

$$\begin{aligned} \partial_t u^p &= \int_{-\infty}^{\infty} \delta(x-y) f(y,t)\,dy + \int_0^t \int_{-\infty}^{\infty} k\frac{\partial^2}{\partial x^2} S(x-y,t-s) f(y,s)\,dy\,ds \\ &= f(x,t)\,dy + k\frac{\partial^2}{\partial x^2} \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) f(y,s)\,dy\,ds = f(x,t) + ku^p_{xx}, \end{aligned}$$

which shows that u^p solves the inhomogeneous heat equation. It is also clear that

$$\lim_{t\to 0} u^p(x,t) = \lim_{t\to 0} \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) f(y,s)\, dy\, ds = 0.$$

Thus, u^p given by (15.5) indeed solves problem (15.3), which finishes the proof that (15.6) solves the original IVP (15.1).

Example 15.1. Find the solution of the inhomogeneous heat equation with the source $f(x,t) = \delta(x-2)\delta(t-1)$ and zero initial data.

Using formula (15.6), and substituting the expression for $f(x,t)$, and $\phi(x) = 0$, we get

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)\delta(y-2)\delta(s-1)\, dy\, ds = \int_0^t S(x-2,t-s)\delta(s-1)\, ds.$$

For the last integral, notice that if $t < 1$, then $\delta(s-1) = 0$ for all $s \in [0,t]$, and if $t > 1$, then the delta function will act on the heat kernel by assigning its value at $s = 1$. Hence,

$$u(x,t) = \begin{cases} 0 & \text{for } 0 < t < 1, \\ S(x-2,t-1) & \text{for } t > 1. \end{cases}$$

This, of course, coincides with our intuition of heat conduction, since the external heat source in this case gives an instantaneous temperature boost to the point $x = 1$ at time $t = 1$. Henceforth, the temperature in the rod will remain zero till the time $t = 1$, and afterward the heat will transfer exactly as in the case of the homogeneous heat equation with data given at time $t = 1$ as $u(x,1) = \delta(x-2)$. □

15.1 Source on the half-line

We will use the reflection method to solve the inhomogeneous heat equation on the half-line. Consider the Dirichlet heat problem

$$\begin{cases} v_t - kv_{xx} = f(x,t), & \text{for } 0 < x < \infty, \\ v(x,0) = \phi(x), \\ v(0,t) = h(t). \end{cases} \tag{15.10}$$

Notice that in the above problem not only the equation is inhomogeneous, but the boundary data is given by an arbitrary function $h(t)$. In this case the Dirichlet condition is called inhomogeneous. We can reduce the above problem to one with zero initial data by the following subtraction method. Defining the new quantity

$$V(x,t) = v(x,t) - h(t), \tag{15.11}$$

we have that

$$\begin{aligned} &V_t - kV_{xx} = v_t - h'(t) - kv_{xx} = f(x,t) - h'(t), \\ &V(x,0) = v(x,0) - h(0) = \phi(x) - h(0), \\ &V(0,t) = v(0,t) - h(t) = h(t) - h(t) = 0. \end{aligned}$$

Thus, $v(x,t)$ solves problem (15.10) if and only if $V(x,t)$ solves the Dirichlet problem

$$\begin{cases} V_t - kV_{xx} = f(x,t) - h'(t), & \text{for } 0 < x < \infty, \\ V(x,0) = \phi(x) - h(0), \\ V(0,t) = 0. \end{cases} \tag{15.12}$$

With this procedure, we essentially combined the heat source given as the boundary data at the endpoint $x = 0$ with the external heat source $f(x,t)$. Notice that $h(t)$ has units of temperature, so its derivative will have units of heat flux, which matches the units of $f(x,t)$. We will denote the combined source in the last problem by $F(x,t) = f(x,t) - h'(t)$, and the initial data by $\Phi(x) = \phi(x) - h(0)$. Since the Dirichlet boundary condition for V is homogeneous, we can extend $F(x,t)$ and $\Phi(x,t)$ to the whole line in an odd fashion, and use the reflection method to solve (15.12). The extensions are

$$\Phi_{\text{odd}}(x) = \begin{cases} \phi(x) - h(0) & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -\phi(-x) + h(0) & \text{for } x < 0, \end{cases} \qquad F_{\text{odd}}(x,t) = \begin{cases} f(x,t) - h'(t) & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -f(-x,t) + h'(t) & \text{for } x < 0. \end{cases}$$

Clearly, the solution to the problem

$$\begin{cases} U_t - kU_{xx} = F_{\text{odd}}(x,t), & \text{for } -\infty < x < \infty, \\ U(x,0) = \Phi_{\text{odd}}(x), \end{cases}$$

is odd, since $U(x,t) + U(-x,t)$ will solve the homogeneous heat equation with zero initial data. Then $U(0,t) = 0$, and the restriction to $x \geq 0$ will solve the Dirichlet problem (15.12) on the half-line. Thus, for $x > 0$,

$$V(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\Phi_{\text{odd}}(y)\,dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)F_{\text{odd}}(y,s)\,dy\,ds.$$

Proceeding exactly as in the case of the (homogeneous) heat equation on the half-line, we will get

$$\begin{aligned} V(x,t) = &\int_0^{\infty} [S(x-y,t) - S(x+y,t)](\phi(y) - h(0))\,dy \\ &+ \int_0^t \int_0^{\infty} [S(x-y,t-s) - S(x+y,t-s)](f(y,s) - h'(s))\,dy\,ds. \end{aligned}$$

Finally, using that $v(x,t) = V(x,t) + h(t)$, we have

$$\begin{aligned} v(x,t) = h(t) + &\int_0^{\infty} [S(x-y,t) - S(x+y,t)](\phi(y) - h(0))\,dy \\ &+ \int_0^t \int_0^{\infty} [S(x-y,t-s) - S(x+y,t-s)](f(y,s) - h'(s))\,dy\,ds. \end{aligned}$$

15.2 Conclusion

Using our intuition of heat conduction as an averaging process with the weight given by the heat kernel, we guessed formula (15.6) for the solution of the inhomogeneous heat equation, treating the inhomogeneity as an external heat source. Employing the propagator operator, this formula coincided exactly with the solution formula for the analogous inhomogeneous ODE, which further hinted at the correctness of the formula. However, to obtain a rigorous proof that formula (15.6) indeed gives the unique solution, we verified that the function given by the formula satisfies the equation and the initial condition by a direct substitution. One can then use this formula along with the reflection method to also find the solution for the inhomogeneous heat equation on the half-line.

Consider the inhomogeneous wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & -\infty < x < \infty, t > 0, \\ u(x,0) = \phi(x), & u_t(x,0) = \psi(x), \end{cases} \tag{16.1}$$

where $f(x,t)$, $\phi(x)$ and $\psi(x)$ are arbitrary given functions. Similar to the inhomogeneous heat equation, the right hand side of the equation, $f(x,t)$, is called the *source* term. In the case of the string vibrations this term measures the external force (per unit mass) applied on the string, and the equation again arises from Newton's second law, in which one now also has a nonzero external force.

As was done for the inhomogeneous heat equation, we can use the superposition principle to break problem (16.1) into two simpler ones:

$$\begin{cases} u^h_{tt} - c^2 u^h_{xx} = 0, \\ u^h(x,0) = \phi(x), \quad u^h_t(x,t) = \psi(x), \end{cases} \tag{16.2}$$

and

$$\begin{cases} u^p_{tt} - c^2 u^p_{xx} = f(x,t), \\ u^p(x,0) = 0, \quad u^p_t(x,t) = 0. \end{cases} \tag{16.3}$$

Obviously, $u = u^h + u^p$ will solve the original problem (16.1). u^h solves the homogeneous equation, so it is given by d'Alambert's formula. Thus, we only need to solve the inhomogeneous equation with zero data, i.e. problem (16.3). We will show that the solution to the original IVP (16.1) is

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds. \tag{16.4}$$

The first two terms in the above formula come from d'Alambert's formula for the homogeneous solution u^h, so to prove formula (16.4), it suffices to show that the solution to the IVP (16.3) is

$$u^p(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds. \tag{16.5}$$

For simplicity, we will seize specifying the superscript and write $u = u^p$ (this corresponds to the assumption $\phi(x) \equiv \psi(x) \equiv 0$, which is the only remaining case to solve).

Recall that we have already solved inhomogeneous hyperbolic equations by the method of characteristics, which we will apply to the inhomogeneous wave equation as well. The change of variables into the characteristic variables and back are given by the following formulas

$$\begin{cases} \xi = x + ct, \\ \eta = x - ct, \end{cases} \qquad \begin{cases} t = \frac{\xi-\eta}{2c}, \\ x = \frac{\xi+\eta}{2}. \end{cases} \tag{16.6}$$

To write the equation in the characteristic variables, we compute u_{tt} and u_{xx} in terms of (ξ,η) using the chain rule.

$$\begin{aligned} u_t &= cu_\xi - cu_\eta, & u_x &= u_\xi + u_\eta, \\ u_{tt} &= c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta}, & u_{xx} &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \end{aligned}$$

so

$$u_{tt} - c^2 u_{xx} = -4c^2 u_{\xi\eta}. \tag{16.7}$$

Notice that $u(x,t) = u(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c})$, and we made an abuse of notation above to identify u with the function $U(\xi,\eta) = u(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c})$. In the same way, we will identify f with the function $F(\xi,\eta) = f(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c})$, and will implicitly understand that the functions in terms of (ξ,η) depend on $(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c})$.

Using (16.7), we can rewrite the inhomogeneous wave equation in terms of the characteristic variables as

$$u_{\xi\eta} = -\frac{1}{4c^2} f(\xi, \eta). \tag{16.8}$$

To solve this equation, we need to successively integrate in terms of η and then ξ. Recall that in previous examples of inhomogeneous hyperbolic equations we performed these integrations explicitly, then changed the variables back to (x, t), and determined the integration constants from the initial conditions. In our present case, however, we would like to obtain a formula for the general function f, so explicit integration is not an option. Thus, to determine the constants of integration, we need to rewrite the initial conditions in terms of the characteristic variables.

Notice that from (16.6), $t = 0$ is equivalent to $(\xi - \eta)/2c = 0$, or $\xi = \eta$. The initial conditions of (16.3) then imply

$$\begin{aligned} &u(\xi, \xi) = 0, \\ &cu_\xi(\xi, \xi) - cu_\eta(\xi, \xi) = 0, \\ &u_\xi(\xi, \xi) + u_\eta(\xi, \xi) = 0, \end{aligned}$$

where the last identity is equivalent to the identity $u_x(x, 0) = 0$, which can be obtained by differentiating the first initial condition of (16.3). From the last two conditions above, it is clear that $u_\xi(\xi, \xi) = u_\eta(\xi, \xi) = 0$, so the initial conditions in terms of the characteristic variables are

$$u(\xi, \xi) = u_\xi(\xi, \xi) = u_\eta(\xi, \xi) = 0. \tag{16.9}$$

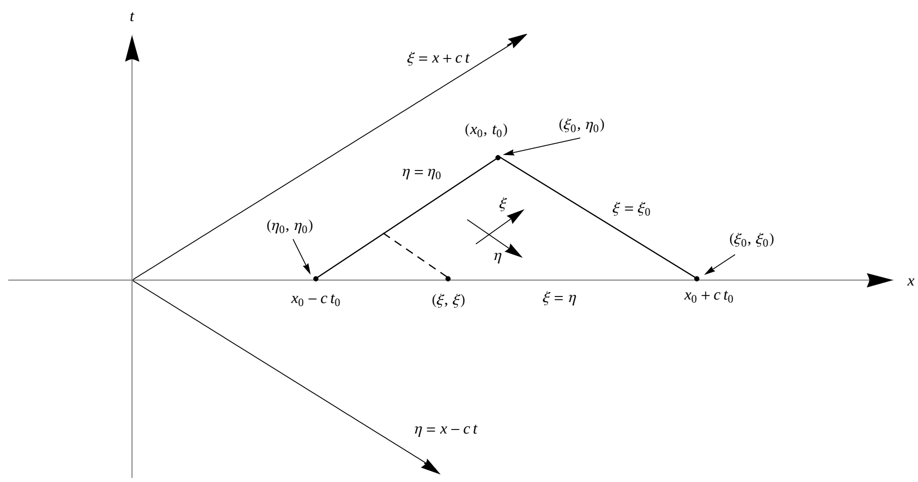

Figure 16.1: The triangle of dependence of the point (x_0, t_0).

Now fix a point (x_0, t_0) for which we will show formula (16.5). This point has the coordinates (ξ_0, η_0) in the characteristic variables. To find the value of the solution at this point, we first integrate equation (16.8) in terms of η from ξ to η_0

$$\int_\xi^{\eta_0} u_{\xi\eta}\, d\eta = -\frac{1}{4c^2} \int_\xi^{\eta_0} f(\xi, \eta)\, d\eta.$$

But

$$\int_\xi^{\eta_0} u_{\xi\eta}\, d\eta = u_\xi(\xi, \eta_0) - u_\xi(\xi, \xi) = u_\xi(\xi, \eta_0)$$

due to (16.9) (this is precisely the reason for the choice of the lower limit), so we have

$$u_\xi(\xi, \eta_0) = \frac{1}{4c^2} \int_{\eta_0}^{\xi} f(\xi, \eta)\, d\eta.$$

Integrating this identity with respect to ξ from η_0 to ξ_0 gives

$$\int_{\eta_0}^{\xi_0} u_\xi(\xi,\eta_0)\,d\xi = \frac{1}{4c^2}\int_{\eta_0}^{\xi_0}\int_{\eta_0}^{\xi} f(\xi,\eta)\,d\eta\,d\xi.$$

Similar to the previous integral,

$$\int_{\eta_0}^{\xi_0} u_\xi(\xi,\eta_0)\,d\xi = u(\xi_0,\eta_0) - u_\xi(\eta_0,\eta_0) = u(\xi_0,\eta_0)$$

due to (16.9). We then have

$$u(\xi_0,\eta_0) = \frac{1}{4c^2}\int_{\eta_0}^{\xi_0}\int_{\eta_0}^{\xi} f(\xi,\eta)\,d\eta\,d\xi = \frac{1}{4c^2}\iint_\triangle f(\xi,\eta)\,d\xi\,d\eta, \tag{16.10}$$

where the double integral is taken over the triangle of dependence of the point (x_0,t_0), as depicted in Figure 16.1. Using the change of variables (16.6), and computing the Jacobian,

$$J = \frac{\partial(\xi,\eta)}{\partial(x,t)} = \begin{vmatrix} 1 & c \\ 1 & -c \end{vmatrix} = -2c,$$

we can transform the double integral in (16.10) to a double integral in terms of the (x,t) variables to get

$$u(x_0,t_0) = \frac{1}{4c^2}\iint_\triangle f(x,t)|J|\,dx\,dt = \frac{1}{2c}\iint_\triangle f(x,t)\,dx\,dt.$$

Finally, rewriting the last double integral as an iterated integral, we will arrive at formula (16.5). This finishes the proof that (16.4) is the unique solution of the IVP (16.1). One can alternatively show that formula (16.4) gives the solution by directly substituting it into (16.1), which is left as a homework problem.

Example 16.1. Solve the inhomogeneous wave IVP

$$\begin{cases} u_{tt} - c^2 u_{xx} = e^x, \\ u(x,0) = u_t(x,0) = 0. \end{cases}$$

Using formula (16.4) with $\phi = \psi = 0$, we get

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} e^y\,dy\,ds = \frac{1}{2c}\int_0^t [e^{x+c(t-s)} - e^{x-c(t-s)}]\,ds \\ &= \frac{e^x}{2c}\left(-\frac{1}{c}e^{c(t-s)}\Big|_0^t - \frac{1}{c}e^{-c(t-s)}\Big|_0^t\right) = \frac{e^x}{2c^2}(e^{ct} + e^{-ct} - 2). \end{aligned}$$

□

16.1 Source on the half-line

Consider the following inhomogeneous Dirichlet wave problem on the half-line

$$\begin{cases} v_{tt} - c^2 v_{xx} = f(x,t), \quad \text{for } 0 < x < \infty, t > 0, \\ v(x,0) = \phi(x), \quad v_t(x,t) = \psi(x), \\ v(0,t) = h(t). \end{cases} \tag{16.11}$$

One can employ the subtraction method that we used for the heat equation to reduce the problem to one with zero Dirichlet data, and then use the reflection method to derive a solution formula for the reduced problem. An alternative simple way, however, is to derive the solution from scratch as follows. Since we know how to find the solution for zero Dirichlet data (use the standard reflection method), we treat the complementary case, that is, assume that the boundary data is nonzero, while $f(x,t) \equiv \phi(x) \equiv \psi(x) \equiv 0$.

From the method of characteristics, we know that the solution can be written as

$$v(x,t) = j(x+ct) + g(x-ct). \tag{16.12}$$

The zero initial conditions then give

$$\begin{aligned} v(x,0) &= j(x) + g(x) = 0, \\ v_t(x,0) &= cj'(x) - cg'(x) = 0, \end{aligned}$$

for $x > 0$. Differentiating the first identity, and dividing the second identity by c, we arrive at the following system for j' and g'

$$\begin{cases} j'(x) + g'(x) = 0, \\ j'(x) - g'(x) = 0, \end{cases} \qquad \Rightarrow \qquad j'(x) = g'(x) = 0.$$

This means that for $s > 0$,

$$j(s) = -g(s) = a$$

for some constant a. On the other hand, the boundary condition for $v(x,t)$ implies

$$v(0,t) = j(ct) + g(-ct) = h(t).$$

But since $ct > 0$, we have $j(ct) = a$, and

$$g(-ct) = h(t) - a, \qquad \text{or} \qquad g(s) = h(-s/c) - a$$

for $s < 0$. Returning to (16.12), notice that the argument of the j term is always positive, so

$$v(x,t) = \begin{cases} a - a & \text{for } x > ct, \\ a + h\left(t - \frac{x}{c}\right) - a & \text{for } x < ct. \end{cases} = \begin{cases} 0 & \text{for } x > ct, \\ h\left(t - \frac{x}{c}\right) & \text{for } x < ct. \end{cases}$$

Thus, for $x > ct$ the solution of (16.11) will be given by (16.4), while for $x < ct$ we have

$$u(x,t) = \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(y)\,dy + h\left(t - \frac{x}{c}\right) + \frac{1}{2c}\iint_D f(y,s)\,dy\,ds,$$

where D is the domain of dependence of the point (x,t).

16.2 Conclusion

The superposition principle was again used to write the solution to the IVP for the inhomogeneous wave equation as a sum of the general homogeneous solution, and the inhomogeneous solution with zero initial data. The inhomogeneous solution was obtained by the method of characteristics through a successive integration in terms of the characteristic variables. One can also derive the solution formula for the inhomogeneous wave equation by simply integrating the equation over the domain of dependence, and using Green's theorem to compute the integral of the left hand side. Yet another way is to approach the solution of the inhomogeneous equation by studying the propagator operator of the wave equation, similar to what we did for the heat equation. These methods are discussed in the appendix.

17 Waves with a source: the operator method

In the previous lecture we used the method of characteristics to solve the initial value problem for the inhomogeneous wave equation,

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & -\infty < x < \infty, t > 0, \\ u(x,0) = \phi(x), & u_t(x,0) = \psi(x), \end{cases} \tag{17.1}$$

and obtained the formula

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds. \tag{17.2}$$

Another way to derive the above solution formula is to integrate both sides of the inhomogeneous wave equation over the triangle of dependence and use Green's theorem.

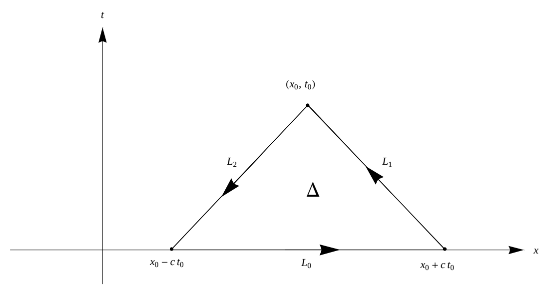

Figure 17.1: The triangle of dependence of the point (x_0, t_0).

Fix a point (x_0, t_0), and integrate both sides of the equation in (17.1) over the triangle of dependence for this point.

$$\iint_{\triangle} (u_{tt} - c^2 u_{xx})\,dx\,dt = \iint_{\triangle} f(x,t)\,dx\,dt. \tag{17.3}$$

Recall that by Green's theorem

$$\iint_D (Q_x - P_t)\,dx\,dt = \oint_{\partial D} P\,dx + Q\,dt,$$

where ∂D is the boundary of the region D with counterclockwise orientation. We thus have

$$\iint_{\triangle} (u_{tt} - c^2 u_{xx})\,dx\,dt = \iint_{\triangle} (-c^2 u_x)_x - (-u_t)_t\,dx\,dt = \oint_{\partial\triangle} -u_t\,dx - c^2 u_x\,dt.$$

The boundary of the triangle of dependence consists of three sides, $\partial\triangle = L_0 + L_1 + L_2$, as can be seen in Figure 17.1, so

$$\iint_{\triangle} (u_{tt} - c^2 u_{xx})\,dx\,dt = \int_{L_0+L_1+L_2} -u_t\,dx - c^2 u_x\,dt,$$

and we have the following relations on each of the sides

$$\begin{aligned} L_0 &: \quad dt = 0 \\ L_1 &: \quad dx = -cdt \\ L_2 &: \quad dx = cdt \end{aligned}$$

Using these, we get

$$\int_{L_0} -c^2 u_x\,dt - u_t\,dx = -\int_{x_0-ct_0}^{x_0+ct_0} u_t(x,0)\,dx = -\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx,$$
$$\int_{L_1} -c^2 u_x\,dt - u_t\,dx = c\int_{L_1} du = c[u(x_0,t_0) - u(0,x_0+ct_0)] = cu(x_0,t_0) - c\phi(x_0+ct_0),$$
$$\int_{L_2} -c^2 u_x\,dt - u_t\,dx = -c\int_{L_2} du = -c[u(0,x_0-ct_0) - u(x_0,t_0)] = cu(x_0,t_0) - c\phi(x_0-ct_0).$$

Putting all the sides together gives

$$\iint_{\triangle} (u_{tt} - c^2 u_{xx})\,dx\,dt = 2cu(x_0,t_0) - c[\phi(x_0+ct_0) + \phi(x_0-ct_0)] - \int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx,$$

and using (17.3), we obtain

$$u(x_0,t_0) = \frac{1}{2}[\phi(x_0+ct_0) + \phi(x_0-ct_0)] + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx + \frac{1}{2c}\iint_{\triangle} f(x,t)\,dx\,dt,$$

which is equivalent to formula (17.2).

17.1 The operator method

For the inhomogeneous heat equation we interpreted the solution formula in terms of the heat propagator, which also showed the parallels between the heat equation and the analogous ODE. We would like to obtain such a description for the solution formula (17.2) as well. For this, consider the ODE analog of the wave equation with the associated initial conditions

$$\begin{cases} \frac{d^2u}{dt^2} + A^2 u = f(t), \\ u(0) = \phi, \quad u'(0) = \psi, \end{cases} \tag{17.4}$$

where A is a constant (a matrix, if we allow u to be vector valued). To find the solution of the inhomogeneous ODE, we need to first solve the homogeneous equation, and then use variation of parameters to find a particular solution of the inhomogeneous equation. The solution of the homogeneous equation is

$$u^h(t) = c_1\cos(At) + c_2\sin(At),$$

and the initial conditions imply that $c_1 = \phi$, and $c_2 = A^{-1}\psi$. To obtain a particular solution of the inhomogeneous equation we assume that c_1 and c_2 depend on t,

$$u^p(t) = c_1(t)\cos(At) + c_2(t)\sin(At),$$

and substitute u^p into the equation to solve for $c_1(t)$ and $c_2(t)$. This procedure leads to

$$c_1(t) = -\int_0^t A^{-1}\sin(As)f(s)\,dt, \qquad c_2(t) = \int_0^t A^{-1}\cos(As)f(s)\,ds.$$

Putting everything together, the solution to (17.4) will be

$$u(t) = \cos(At)\phi + A^{-1}\sin(At)\psi + \int_0^t A^{-1}\sin(A(t-s))f(s)\,ds.$$

If we now define the propagator

$$\mathscr{S}(t)\psi = A^{-1}\sin(At)\psi,$$

then the solution to (17.4) can be written as

$$u(t) = \mathscr{S}'(t)\phi + \mathscr{S}(t)\psi + \int_0^t \mathscr{S}(t-s)f(s)\,ds. \tag{17.5}$$

For the wave equation, similarly denoting the operator acting on ψ from d'Alambert's formula by

$$\mathscr{S}(t)\psi = \int_{x-ct}^{x+ct} \psi(y)\,dy,$$

we can rewrite formula (17.2) in exactly the same form as (17.4). The moral of this story is that having solved the homogeneous equation and found the propagator, we have effectively derived the solution of the inhomogeneous equation as well. The rigorous connection between the solution of the homogeneous equation and that of the inhomogeneous wave equation is contained in the following statement.

Duhamel's Principle. Consider the following 1-parameter family of wave IVPs

$$\begin{cases} u_{tt}(x,t;s) - c^2 u_{xx}(x,t;s) = 0, \\ u(x,s;s) = 0, \quad u_t(x,s;s) = f(x,s), \end{cases} \tag{17.6}$$

then the function

$$v(x,t) = \int_0^t u(x,t;s)\,ds$$

solves the inhomogeneous wave equation with vanishing data, i.e.

$$\begin{cases} v_{tt}(x,t) - c^2 v_{xx}(x,t) = f(x,t), \\ v(x,0) = 0, \quad v_t(x,0) = 0. \end{cases}$$

Note that the initial conditions of the s-IVP (17.6) are given at time $t = s$, and the initial velocity is $\psi(x;s) = f(x,s)$. Duhamel's principle has the physical description of replacing the external force by its effect on the velocity. From Newton's second law, the force is responsible for acceleration, or change in velocity per unit time. So if we can account for the effect of the external force on the instantaneous velocity, then the the solution of the equation with the external force can be found by solving the homogeneous equations with the effected velocities, namely (17.6), and "summing" these solutions over the instances $t = s$.

We prove Duhamel's principle by direct substitution. The derivatives of v are

$$\begin{aligned} v_t(x,t) &= u(x,t;t) + \int_0^t u_t(x,t;s)\,ds = \int_0^t u_t(x,t;s)\,ds, \\ v_{tt}(x,t) &= u_t(x,t;t) + \int_0^t u_{tt}(x,t;s)\,ds = f(x,t) + \int_0^t u_{tt}(x,t;s)\,ds, \\ v_{xx}(x,t) &= \int_0^t u_{xx}(x,t;s)\,ds, \end{aligned}$$

where we used the initial conditions of (17.6). Substituting this into the wave equation gives

$$(\partial_t^2 - c^2\partial_x^2)v = \int_0^t [u_{tt}(x,t;s) - c^2 u_{xx}(x,t;s)]\,ds + f(x,t) = f(x,t),$$

and v indeed solves the inhomogeneous wave equation. It is also clear that v has vanishing initial data.

Duhamel's principle gives an alternative way of proving that (17.2) solves the inhomogeneous wave equation. Indeed, from d'Alambert's formula for (17.6) and a time shift $t \mapsto t - s$, we have

$$u(x,t;s) = \frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\, dy.$$

Thus, the solution of the inhomogeneous wave equation with zero initial data is

$$v(x,t) = \int_0^t u(x,t;s)\, ds = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\, dy\, ds.$$

17.2 Conclusion

We defined the wave propagator as the operator that maps the initial velocity to the solution of the homogeneous wave equation with zero initial displacement. Using this operator, the solution of the inhomogeneous wave equation can be written in exactly the same form as the solution of the analogous inhomogeneous ODE in terms of its propagator. The significance of this observation is in the connection between the solution of the homogeneous and that of the inhomogeneous wave equations, which is the substance of Duhamel's principle. Hence, to solve the inhomogeneous wave equation, all one needs is to find the propagator operator for the homogeneous equation.