# MATH 425 LECTURE 1: Review of ODEs.

→ Syllabus on Canvas

## Chapter 0: ODEs

An ODE relates a function of one variable, $y(x)$, with its derivatives $y'(x), y''(x), \dots$ The <u>order</u> of the ODE is the order of the highest derivative.

We will review $1^{st}$ and $2^{nd}$ order ODE.

### 0.1 First-order ~~ODEs~~ ODEs.

In general, they can be written as $F(x, y, y') = 0$.

Little can be said in such generality.

Most times, there is only one solution $y(x)$ passing through a given point. That is,

$$\left.\begin{array}{l} F(x,y,y')=0 \\ y(a)=b \end{array}\right\} \text{ normally has only one solution.}$$

Initial value problem

• We want to review separable and linear equations.

<u>Separable ODE</u>: $y'(x) = f(x)\,g(y)$ for given functions $f, g$.

Solution: $\dfrac{dy}{g(y)} = f(x)\,dx \rightsquigarrow \displaystyle\int \frac{dy}{g(y)} = \int f(x)\,dx + C$.

<u>Ex</u>: $\left.\begin{array}{l} y'(x) = x\,y^2 \\ y(0) = 1 \end{array}\right\}$

<u>Sol</u>: $\dfrac{dy}{y^2} = x\,dx \rightsquigarrow \dfrac{-1}{y(x)} = \dfrac{x^2}{2} + C \underset{y(0)=1}{\rightsquigarrow} -1 = C$

So, $$y(x) = \frac{-2}{x^2 - 2} \quad //$$

<u>Ex</u>: $x'(t) + x^2(t)\cos(t) = 0$

<u>Sol</u>: $\dfrac{dx}{x^2} = -\cos(t)\,dt \rightsquigarrow \dfrac{-1}{x} = -\sin(t) + C \rightsquigarrow x(t) = \dfrac{1}{\sin(t) - C}$

## Linear ODEs

We can think of functions as vectors (we have indeed the addition and scalar multiplications).

Matrices? $\rightsquigarrow$ Linear transformation.

A transformation or operator $\mathcal{L}$ is linear if

1) $\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$
2) $\mathcal{L}(cf) = c\mathcal{L}(f)$

for all functions $f, g$ and scalars $c$.

__Remark__: One can check that if $\mathcal{L}$ is linear, then

- $\mathcal{L}(0) = 0$
- $\mathcal{L}(\alpha f + \beta g) = \alpha \mathcal{L}(f) + \beta \mathcal{L}(g)$.

__Ex__: Are the following linear operators?

1) Multiplication by a fixed function: $\mathcal{L}(f) = h(x)f(x)$ ✓

2) Differentiation: $\mathcal{L}(f) = D(f)$ ✓

3) $\mathcal{L}(f) = f(x)^2$ ×

4) $\mathcal{L}(f) = 1 + f(x)$ ×

- We call first-order linear differential operator to

$$\mathcal{L}(f) = a(x)\,D(f) + b(x)\,f$$

and thus a linear first-order differential equation is of the form

$$\mathcal{L}(u) = h(x) \quad \text{(with } h(x) \text{ given).}$$

How to solve $\mathcal{L}(u) = h(x)$?

I.e. find $u(x)$ / $a(x)u'(x) + b(x)u(x) = h(x)$ ?

<u>Ex</u>: $u'(x) + u(x) = \sin(x)$

<u>Sol</u>: Note that $u'(x) + u(x) = e^{-x}\frac{d}{dx}\left(e^{x}u(x)\right)$, thus

$(e^{x}u(x))' = e^{x}\sin(x) \rightsquigarrow$

$\rightsquigarrow e^{x}u(x) = \int e^{x}\sin(x)\,dx + C \rightsquigarrow u = e^{-x}\left(\int e^{x}\sin(x)\,dx + C\right)$

- In general: $a(x)u' + b(x)u = h(x) \Longrightarrow u'(x) + p(x)u(x) = g(x)$.

1) $u'(x) + p(x)u(x) = e^{-\int p(x)}\frac{d}{dx}\left(e^{\int p(x)}u(x)\right)$

2) $\frac{d}{dx}\left(e^{\int p(x)}u(x)\right) = e^{\int p(x)}g(x) \rightsquigarrow u(x) = e^{-\int p(x)}\left(\int e^{\int p(x)}g(x)\,dx + C\right)$

Ex: $\left.\begin{array}{l} u' + \frac{1}{x} u = x^2 \\ u(1) = 1 \end{array}\right\}$

Sol: $u(x) = e^{-\int \frac{1}{x}} \left( \int e^{\int \frac{1}{x}} x^2 \, dx + c \right) = e^{-\log x} \left( \int x^3 \, dx + c \right) =$

$= \frac{1}{x} \left( \frac{x^4}{4} + c \right)$

$u(1) = 1 = \frac{1}{4} + c \Rightarrow c = \frac{3}{4} \Rightarrow u(x) = \frac{x^3}{4} + \frac{3}{4x}$ //.

→ Uniqueness

Theorem: $\exists!$ sol. to $\left.\begin{array}{l} u' + p(x) u = g(x) \\ u(a) = b \end{array}\right\}$

Proof: Let $u_1(x), u_2(x)$ be sols, $v(x) = u_1(x) - u_2(x)$.

Note that $\left.\begin{array}{l} v' + p(x) v = 0 \\ v(a) = 0 \end{array}\right\} (*)$

Let $v(x)$ be any solution of $(*)$, and consider $w(x) = e^{\int p(x)} v(x)$.

Then, $w'(x) = e^{\int p(x)} v'(x) + e^{\int p(x)} p(x) v(x) =$

$= e^{\int p(x)} \left( v'(x) + p(x) v(x) \right) = 0 \Rightarrow$

$\Rightarrow w(x) = \text{cte}$; $w(a) = e^{\int p(a)} v(a) = 0 \Rightarrow w \equiv 0 \Rightarrow v \equiv 0$ //.

## $\boxed{0.2}$ Second-order ODEs.

We will only consider <u>homogeneous linear</u> ODEs with <u>constant coeff</u>.

That is,

$$u''(x) + bu'(x) + cu(x) = 0 \qquad (b, c \in \mathbb{R} \text{ cts}).$$

$\hookrightarrow \underbrace{\mathcal{L}(u) = 0}$, $\mathcal{L} = D^2 + bD + cI$.

We are finding the <u>nullspace</u> of the linear operator $\mathcal{L}$.

$\left[\, A\vec{x} = \vec{0} \sim \text{i.e. eigenvectors with } \lambda = 0 \,\right]$

"Eigenfunctions": $\mathcal{L}(u) = \lambda u$

Note that: 1) $D(e^{rx}) = re^{rx} \rightsquigarrow e^{rx}$ eigenvector of $D$ with eigenvalue $r$.

2) ~~$\mathcal{L}(e^{rx}) = r$~~

Think of $\mathcal{L}(u)$ as $\mathcal{L}(u) = p(D)$ with $p(x) = x^2 + bx + c$.

Then,

$$\mathcal{L}(e^{rx}) = r^2 e^{rx} + bre^{rx} + ce^{rx} = p(r)e^{rx} \Rightarrow$$

$\Rightarrow e^{rx}$ is eigenvector of $\mathcal{L}$ with eigenvalue $p(r)$.

Nullspace of $\mathcal{L} \rightarrow$ find eigenvectors for $p(r) = 0$.

So we obtain three cases:

1) Two roots of $p$ are distinct real numbers $r_1, r_2$:

$$\mathcal{L}(u)=0 \to u(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

2) Two complex (conjugates) roots: real part $-\frac{b}{2}$, imaginary $\overbrace{\frac{1}{2}\sqrt{4c-b^2}}^{\alpha}$

$$\hookrightarrow u(x) = e^{-b\frac{x}{2}}\left(c_1 \cos(\alpha x) + c_2 \sin(\alpha x)\right)$$

3) Two real repeated: $r_1 = r_2 = \frac{-b}{2}$.

$$\hookrightarrow u(x) = c_1 e^{-b\frac{x}{2}} + \underbrace{c_2 x e^{-b\frac{x}{2}}}_{?}$$

$$\to r_1 \text{ repeated} \to \begin{cases} p(r_1)=0 \\ \frac{dp}{dr}(r)\Big|_{r_1} = 0 \end{cases} \to \frac{d}{dr}\left(p(r)e^{rx}\right)\Big|_{r_1} = 0 \to$$

$$\to \frac{d}{dr}\left(L(e^{rx})\right)\Big|_{r=r_1} = 0 \to \frac{d}{dr}L(e^{rx})\Big|_{r=r_1} = L\left(\frac{d}{dr}e^{rx}\right)\Big|_{r=r_1} = L\left(x e^{rx}\right)\Big|_{r=r_1} = 0 \quad //.$$

<u>Remark</u>: We could solve $\mathcal{L}(u)=0$ as a system:

$$\begin{matrix} u_1(x) = u(x) \\ u_2(x) = u'(x) \end{matrix} \cdots \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix}' = A\begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix} \to \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix} = e^{Ax}\begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix}$$

and also we can solve a $1^{st}$-order system as a second order equation.

• Uniqueness

1) Initial value problem: $\left.\begin{aligned} u'' + cu &= 0 \\ u(0) &= a \\ u'(0) &= b \end{aligned}\right\} \quad c > 0$

Proof: Let $u_1, u_2$ be two sol. Denote $v = u_1 - u_2$.

$$\left.\begin{aligned} v'' + cv &= 0 \\ v(0) = 0 &= v'(0) \end{aligned}\right\}$$

Multiply the eq. by $v'$: ~~$v'' + cv^2 = 0$~~

$$v'v'' + cvv' = \frac{1}{2}\frac{d}{dx}(v'(x))^2 + \frac{c}{2}\frac{d}{dx}(v(x))^2 = 0 \Rightarrow$$

$$\Rightarrow (v')^2 + c(v)^2 = \text{cte}.$$

Since $v'(0) = v(0) = 0 \Rightarrow \underbrace{v'^2 + cv^2}_{\text{"energy"}} = 0 \Rightarrow v \equiv 0$ //.

2) Boundary value problem:

Ex: $\left.\begin{aligned} u'' + u &= 0 \\ u(0) &= 0 \\ u(\pi) &= 0 \end{aligned}\right\}$ $\qquad \left.\begin{aligned} u'' + u &= 0 \\ u(0) &= 0 \\ u(\pi) &= 1 \end{aligned}\right\}$ No solution

$u = c_1 \cos x + c_2 \sin x$ ~~no solution~~ ~~inf. sol.~~ ~~inf. solution~~ ~~no solution~~

Inf. solution.

Ex: $\left.\begin{array}{l} u'' - cu = 0 \\ u(a) = p \\ u(b) = q \end{array}\right\}$ $(c \geq 0)$ (uniqueness)

Sol: $v = u_1 - u_2 \Rightarrow \left.\begin{array}{l} v'' - cv = 0 \\ v(a) = 0 = v(b) \end{array}\right\}$

Multiply by $v$: $v''v - cv^2 = 0$ and integrate between $a, b$:

$$\int_a^b v''(x)v(x)\,dx = c\int_a^b v(x)^2\,dx$$

Integrate by parts: $\begin{array}{ll} u = v(x) & du = v'(x)\,dx \\ dv = v''(x)\,dx & v = v'(x) \end{array}$

$$\int_a^b v''(x)\,v(x)\,dx = v'(x)\,v(x)\Big]_a^b - \int_a^b (v'(x))^2\,dx\,.$$

so $\displaystyle\int_a^b v(x)^2\,dx = \frac{-1}{c}\int_a^b (v'(x))^2\,dx \leq 0 \Rightarrow \int_a^b v(x)^2\,dx = 0 \Rightarrow$

$\Rightarrow v \equiv 0$.