HOMEWORK ASSIGNMENT 4

Name: **Due:** Friday March 15, 4PM

All the problems in this homework are from W. Strauss book. The estimated level of difficulty is marked as *,**,***.

Problem 1*: Strauss, Section 2.4 #1, p.52

Solve the diffusion equation with the initial condition

$$\phi(x) = 1 \quad \text{for } |x| < l \qquad \text{and} \qquad \phi(x) = 0 \quad \text{for } |x| > l.$$

Write your answer in terms of $\mathrm{Erf}(x)$.

Problem 2*: Strauss, Section 2.4 #17, p.54

Solve the diffusion equation with variable dissipation:

$$u_t - ku_{xx} + bt^2 u = 0 \quad \text{for } -\infty < x < \infty \quad \text{with } u(x,0) = \phi(x),$$

where $b > 0$ is a constant. Hint: The solutions of the ODE $w_t + bt^2 w = 0$ are $Ce^{-bt^3/3}$. So make the change of variables $u(x,t) = e^{-bt^3/3} v(x,t)$ and derive an equation for v.

Problem 3**: Strauss, Section 2.4 #19, p.54

(a) Show that $S_2(x,t,t) = S(x,t)S(y,t)$ satisfies the two-dimensional diffusion equation: $S_t = k(S_{xx} + S_{yy})$.

(b) Deduce that $S_2(x,y,t)$ is the source function for two-dimensional diffusions (i.e., that the general solution is given as convolutions with this source).

Problem 4**: Strauss, Section 3.1 #1, p.60

Solve $u_t = ku_{xx}$, $u(x,0) = e^{-x}$, $u(0,t) = 0$ on the half-line $0 < x < \infty$.

Problem 5***: Strauss, Section 3.1 #4, p.60

Consider the following problem with a Robin boundary condition:

$$\begin{aligned} u_t = ku_{xx} \qquad & 0 < x < \infty, t > 0, \\ u(x,0) = x \qquad & \text{for } t = 0, 0 < x < \infty, \\ u_x(0,t) - 2u(0,t) = 0, \qquad & x = 0. \end{aligned} \tag{1}$$

The purpose of this exercise is to verify the solution formula for the problem above. Let $f(x) = x$ for $x > 0$, let $f(x) = x + 1 - e^{2x}$ for $x < 0$, and let

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y) dy.$$

(a) What PDE and initial condition does $v(x,t)$ satisfy for $-\infty < x < \infty$?

(b) Let $w = v_x - 2v$. What PDE and initial condition does $w(x,t)$ satisfy for $-\infty < x < \infty$?

(c) Show that $f'(x) - 2f(x)$ is an odd function.

(d) Show that w is an odd function of x (use the equation that defines w and uniqueness).

(e) Deduce that $v(x,t)$ satisfies (1) for $x > 0$. Assuming uniqueness, deduce that the solution of (1) is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y).$$

Problem 6**: Strauss, Section 3.3 #2, p.71

Solve the inhomogeneous diffusion problem on the half-line

$$\begin{aligned} &v_t - kv_{xx} = f(x,t) && \text{for } 0 < x < \infty,\ 0 < t < \infty, \\ &v(0,t) = h(t), && v(x,0) = \phi(x). \end{aligned}$$

Problem 7**: Strauss, Section 3.3 #3, p.71

Solve the inhomogeneous Neumann diffusion problem on the half-line

$$\begin{aligned} &w_t - kw_{xx} = f(x,t) && \text{for } 0 < x < \infty,\ 0 < t < \infty, \\ &w_x(0,t) = h(t), && w(x,0) = \phi(x). \end{aligned}$$

Problem 8*: Strauss, Section 3.4 #3, p.79

Solve $u_{tt} = c^2 u_{xx} + \cos x$, $u(x,0) = \sin x$, $u_t(x,0) = 1 + x$.

Problem 9***: Strauss, Section 3.4 #5, p.79

Let $f(x,t)$ be any function and let $u(x,t) = \frac{1}{2c}\iint_\Delta f$, where Δ is the triangle of dependence. Verify directly by differentiation that

$$u_{tt} = c^2 u_{xx} + f \text{ and } u(x,0) \equiv u_t(x,0) \equiv 0.$$

Hint: Begin by writing the formula as the iterated integral

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-ct+cs}^{x+ct-cs} f(y,s)\,dy\,ds$$

and differentiate *with care* (the limits of integration depend also on x, t).

Problem 10*: Strauss, Section 4.1 #4, p.89

Consider waves in a resistant medium that satisfy the problem

$$\begin{aligned} u_{tt} = c^2 u_{xx} - r u_t \qquad & \text{for } 0 < x < l, \\ u = 0 \qquad & \text{at both ends,} \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), & \end{aligned} \tag{2}$$

where r is a constant, $0 < r < 2\pi c/l$. Write down the series expansion of the solution.

Problem 11*: Strauss, Section 4.1 #6, p.89

Separate the variables for the equation $tu_t = u_{xx} + 2u$ with the boundary conditions $u(0,t) = u(\pi,t) = 0$. Show that there are an infinite number of solutions that satisfy the initial condition $u(x,0) = 0$. So uniqueness is false for this equation.

Problem 12:

Read Chapter 4 of W. Strauss book. Which part was most confusing to you?