Extra Problems for Midterm 2

Problem 1

Suppose you know that the eigenvalues of a 3×3 matrix A are $0, 1, 2$ with corresponding eigenvectors $\vec{v}_0, \vec{v}_1$, and $\vec{v}_2$.

1. What are the eigenvectors and eigenvalues of A^2?
2. What are the eigenvectors and eigenvalues of $A + Id$?
3. Is A invertible? If so what are the eigenvectors and eigenvalues of A^{-1}?

Problem 2

Find the coefficients for the model below that best fit the data $x = 0, -\pi/2, \pi/2, \pi$, $y = 3, -1, 1 - 5$, $z = 1/2, 1, 2, 3$ in the least squares sense:

$$z = ax + by \sin(x).$$

Problem 3

We want to find the coefficients a, b, c of a model $z = a + bx + cy$. We are given the following data points: $x = 1, -1, -1, 1$, $y = 1, -1, 1, -1$ and $z = 1, 2, 3, 4$.

1. Write down in *matrix form* the four equations that would be satisfied if the plane went through all 4 points. What is special about the columns of the matrix?
2. Find the coefficients a, b, c that minimize the least square error.

Problem 4

A subspace V of $\mathbb{R}^3$ is spanned by the columns of

$$\begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 1 & 1 \end{bmatrix}.$$

1. Apply the Gram-Schmidt process to find two orthonormal vectors $\vec{q}_1$, $\vec{q}_2$ which also span V.
2. Find an orthogonal matrix Q so that QQ^T is the matrix which orthogonally projects vectors onto V.
3. Find the least squared error solution to the linear system

$$Q \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Problem 5

1. Apply Gram-Schmidt algorithm to find an orthonormal basis $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$ with the same span as

$$\{\vec{a}_1, \vec{a}_2, \vec{a}_3\} = \{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \}.$$

2. Find the projection of $\vec{b} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ onto the subspace spanned by $\vec{a_1}$, $\vec{a_2}$ and $\vec{a_3}$.

Problem 6

1. Find an orthonormal basis (using Gram-Schmidt method) for the subspace S in $\mathbb{R}^4$ spanned by all solutions of

$$x_1 + x_2 + x_3 - x_4 = 0.$$

2. Find an orthonormal basis for the orthogonal complement $S^\perp$.
3. Find $\vec{b}_1$ in S and $\vec{b}_2$ in $S^\perp$ so that $\vec{b}_1 + \vec{b}_2 = (1, 1, 1, 1)$.
4. Find the point in S closest to $(1, 0, 0, 0)$.

Problem 7

Consider the planes $H = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ and $V = \{(x, y, z) \in \mathbb{R}^3 : z = y\}$. We will use the following basis for H:

$$\mathcal{E} = \{\vec{e}_1, \vec{e}_2\} = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \}.$$

1. Write the vector $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ in the basis $\mathcal{E}$. That is, give the coordinates of $\vec{u}$ in $\mathcal{E}$.
2. Find an orthonormal basis $\mathcal{V}$ for V.
3. Write the vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in the basis $\mathcal{V}$.

4. The projection of vectors in V onto H is a linear transformation. Find the matrix of this linear transformation using the basis $\mathcal{V}$ and $\mathcal{E}$.
5. Find the area of the triangle with vertex $(0,0,0)$, $(1,1,1)$ and $(0,1,1)$.

Problem 8

The polynomials $\vec{u}_1 = 1$, $\vec{u}_2 = x - 2$, and $\vec{u}_3 = (x-2)^2$ form a basis for the space of (at most) quadratic polynomials in x, as do the polynomials $\vec{v}_1 = 1$, $\vec{v}_2 = x + 1$, and $\vec{v}_3 = (x+1)^2$. Find the change of basis matrix from $\{u_i\}$ to $\{v_i\}$ and use it to find numbers a, b, c such that $-1(x-2) + 3(x-2)^2 = a + b(x+1) + c(x+1)^2$.

Problem 9

Find the limit of A^k as k goes to infinity for

$$A = \begin{bmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{bmatrix}$$

Problem 10

Let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & b \end{bmatrix}$$

For which values of b does A have distinct eigenvalues?

Problem 11

Let P be a matrix that projects vectors of $\mathbb{R}^3$ onto the plane $z = 0$. What are the eigenvalues and eigenvectors of P?

Problem 12

Consider the linear differential system

$$\begin{aligned} x' &= x + 3y \\ y' &= 2x + 2y. \end{aligned}$$

1. For which matrix A can we rewrite this system as $\begin{bmatrix} x \\ y \end{bmatrix}' = A \begin{bmatrix} x \\ y \end{bmatrix}$?
2. Find an invertible matrix S and a diagonal matrix D so that $A = SDS^{-1}$.
3. Write the exponential matrix e^{At}

4. Find the solution $x(t)$ and $y(t)$ to this linear differential system subject to the initial conditions $x(0) = -5$ and $y(0) = 5$.

5. If $x(t)$ and $y(t)$ represent two species that have a mutually symbiotic relationship, say $x(t)$ number of flowers and $y(t)$ number of bees, how many bees per flowers are there in the equilibrium situation (that is, as time goes to infinity)?

Problem 13

Consider the linear differential system

$$x' = 3x - 4y$$
$$y' = 2x - 3y.$$

1. For which matrix A can we rewrite this system as $\begin{bmatrix} x \\ y \end{bmatrix}' = A \begin{bmatrix} x \\ y \end{bmatrix}$?

2. Find an invertible matrix S and a diagonal matrix D so that $A = SDS^{-1}$.

3. Find the exponential matrix e^{At}.

4. Find the solution $x(t)$ and $y(t)$ to this linear differential system subject to the initial conditions $x(0) = 1$ and $y(0) = 0$. What is the ratio $y(t)/x(t)$ as t goes to infinity?

5. For what initial conditions $x(0)$, $y(0)$ does the solution $(x(t), y(t))$ to this differential system lie on a single straight line in $\mathbb{R}^2$ for all t? (Hint: You can do this explicitly, as in d), or just thinking on the phase portrait)

True or False

1. If A is a square matrix and B is obtained from A via row operation R2'=R2+3R1, then B has the same eigenvalues as A.

2. If A is invertible and has one eigenvalue λ, then $1/\lambda$ is an eigenvalue of A^{-1}.

3. If Q is an orthogonal matrix, then the corresponding linear transformation preserves lengths and angles, i.e., length of $Q\vec{x}$ is equal to length of $\vec{x}$ and the angle between $\vec{x}$ and $\vec{y}$ is equal to the angle between $Q\vec{x}$ and $Q\vec{y}$.

4. A 3 by 3 matrix with eigenvalues $0, 0, 1$ always has rank(A)=1.

5. A basis for eigenvectors for nonzero eigenvalues of A is a basis for $C(A)$ for any matrix A.

6. The only upper triangular 3×3 matrix with 1s on the diagonal which is diagonalizable is the identity matrix.

7. If A is an orthogonal matrix, then $\lambda = 2$ cannot be an eigenvalue.
8. For any square matrix A, $\det(2A) = 2\det(A)$.