Math 425 / AMCS 525 **Dr. DeTurck**
Practice problems for midterm 2 **March 23, 2016**

1. Say that a function is *oddly odd* if it satisfies both the conditions

$$f(-x) = -f(x), \qquad f(L+x) = f(L-x)$$

(a) Show that such a function is periodic with period $4L$.

(b) Draw the graph of a non-zero oddly odd function for $-5L \le x \le 5L$ (pick one that is interesting but not too interesting, perhaps have the graph consist mostly of line segments). What (if any) kind of symmetry does it have around the line $x = L$? ...around the line $x = 2L$?

(c) Show that the Fourier series of an oddly odd function is of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{(2n-1)\pi x}{2L}.$$

Give a formula for the coefficients b_n.

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2. Let

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 < x < 2 \end{cases}$$

Solve the heat equation $u_t = u_{xx}$ for $x \in [0,2]$ and $t \in [0,\infty)$ with initial condition $u(x,0) = f(x)$ and boundary conditions $u(0,t) = u(2,t) = 0$. Draw a sketch of the graph of $u(x,\epsilon)$ for a fixed, very small value of ϵ and $0 \le x \le 2$.

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3. (a) Find the Fourier (cosine) series of the function $f(x) = x^2$, $-\pi < x < \pi$.

(b) Draw the graph of the function to which your series converges. Explain how you know the series converges pointwise to this function. Does it converge uniformly?

(c) Use the series to show that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots = \frac{\pi^2}{6}$$

and

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots + \frac{(-1)^{n+1}}{n^2} + \cdots = \frac{\pi^2}{12}$$

(d) Use the results in part (c) to deduce

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n-1)^2} + \cdots = \frac{\pi^2}{8}$$

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4. Solve the initial-boundary value problem for the wave equation:

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < 1, \quad t > 0$$

where $u(x,0) = \sin \pi x$, $u_t(x,0) = 0$, $u(0,t) = 0$, $u(1,t) = 1$.

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5. (a) Find the eigenvalues and eigenfunctions of the boundary-value problem:

$$u'' + \lambda u = 0, \qquad u(0) = 0, \qquad u'(3) + u(3) = 0$$

for $u(x)$ defined on the interval [0,3].

(b) If we number the eigenvalues in increasing order, so that $\lambda_1 < \lambda_2 < \lambda_3 < ...$, find A and B so that

$$\lim_{n\to\infty} \left(\lambda_n - (An + B)^2\right) = 0.$$

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6. This problem shouldn't require any integration, but (b) and especially (c) will require some thinking.

(a) Solve the Laplace equation

$$u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = 0$$

on the inside of the disk $r < 2$ with boundary condition

$$u(2,\theta) = 8\sin(3\theta)$$

for $0 < \theta < 2\pi$.

(b) Solve the Laplace equation on the *outside* of the circle $r = 1$ (that is, for $r > 1$) with boundary condition

$$u(1,\theta) = \sin(2\theta).$$

Assume we want the solution to remain bounded as $r \to \infty$. How does this change the form of the solution?

(c) Solve the Laplace equation in the *annulus* inside the circle $r = 2$ but outside the circle $r = 1$, i.e., for $1 < r < 2$ with boundary conditions

$$u(1,\theta) = \sin(2\theta) \qquad u(2,\theta) = 8\sin(3\theta).$$

Since there is neither a condition at $r = 0$ nor at infinity, both parts of $R(r)$ in the separated solutions come into play.

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7. For the exam, be sure you know the integrals:

$$\int \cos ax \, \cos bx \, dx, \qquad \int \sin ax \, \cos bx \, dx.$$