MATH 312 LECTURE 2	: Matrices and inverses.

1) Systems in matrix form.

We can write systems, like the following one

$$
\begin{aligned}
2x - y + z &= 2 \quad & E1\\
4x + y - z &= 1 \quad & E2\\
2x + 2y + 2z &= 1 \quad & E3
\end{aligned}
$$

using the so-called augmented matrix:

$$
\begin{array}{c c}
 & \begin{array}{cccc} x & y & z & \end{array} \\
\begin{array}{c} E1 \\ E2 \\ E3 \end{array} &
\left[\begin{array}{ccc|c}
\boxed{2} & -1 & 1 & 2 \\
4 & 1 & -1 & 1 \\
2 & 2 & 2 & 1
\end{array}\right]
\end{array}
$$

Now, the gaussian elimination steps are row operations:

$$
\begin{aligned}
R2' &= R2 - 2R1\\
R3' &= R3 - R1
\end{aligned}
\rightarrow
\left[\begin{array}{ccc|c}
\boxed{2} & -1 & 1 & 2 \\
0 & \boxed{3} & -3 & -3 \\
0 & 3 & 1 & -1
\end{array}\right]
\xrightarrow{R3'' = R3' - R2'}
\left[\begin{array}{ccc|c}
\boxed{2} & -1 & 1 & 2 \\
0 & \boxed{3} & -3 & -3 \\
0 & 0 & \boxed{4} & +2
\end{array}\right]
$$

Therefore: $z = \frac{+2}{4} = \frac{+1}{2}$

$3y = -3 + 3z \rightarrow y = -1 + \frac{1}{2} = \frac{-1}{2}$

$2x = 2 + y - z = 2 - \frac{1}{2} - + \frac{1}{2} = 1$

$x = 1/2$

$y = -1/2$

$z = +1/2$

The matrix obtained after applying gaussian elimination is said to be in "echelon form".

• Let's see one example with infinite solutions:

$$\left.\begin{aligned} 2x_1 - x_2 + x_3 + x_4 &= 2 \\ 4x_1 + x_2 - x_3 + 2x_4 &= 1 \\ 2x_1 + 2x_2 + 2x_3 - x_4 &= 1 \end{aligned}\right\}$$

$$\left[\begin{array}{cccc|c} 2 & -1 & 1 & 1 & 2 \\ 4 & 1 & -1 & 2 & 1 \\ 2 & 2 & 2 & -1 & 1 \end{array}\right] \xrightarrow[R3' = R3 - R1]{R2' = R2 - 2R1} \left[\begin{array}{cccc|c} \boxed{2} & -1 & 1 & 1 & 2 \\ 0 & \boxed{3} & -3 & 0 & -3 \\ 0 & 3 & 1 & -2 & -1 \end{array}\right] \xrightarrow{R3'' = R3' - R2'}$$

$$\left[\begin{array}{cccc|c} \boxed{2} & -1 & 1 & 1 & 2 \\ 0 & \boxed{3} & -3 & 0 & -3 \\ 0 & 0 & \boxed{4} & \underset{\substack{\uparrow \\ \text{free variable}}}{-2} & 2 \end{array}\right]$$

Let $x_4 = \alpha \in \mathbb{R}$ parameter. Then,

$4x_3 = 2 + 2\alpha \Rightarrow x_3 = \frac{1}{2} + \frac{\alpha}{2}$

$3x_2 = -3 + 3x_3 \Rightarrow x_2 = -1 + \frac{1}{2} + \frac{\alpha}{2} = \frac{-1}{2} + \frac{\alpha}{2}$

$2x_1 = 2 + x_2 - x_3 - \alpha \Rightarrow x_1 = 1 + \frac{1}{2}\left(\frac{-1}{2} + \frac{\alpha}{2}\right) - \frac{1}{2}\left(\frac{1}{2} + \frac{\alpha}{2}\right) - \alpha =$

$= 1 - \frac{1}{4} - \frac{1}{4} + \frac{\alpha}{4} - \frac{\alpha}{4} = \frac{1}{2}$

That is, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 1 \end{bmatrix}$

2) Gauss-Jordan elimination

This method consists in continuing elimination upwards after we find the "echelon form":

Ex:
$$\left.\begin{aligned} x_1 - x_3 + 4x_4 &= -1 \\ x_1 + x_2 + x_3 &= 1 \\ x_1 + x_2 + 2x_3 - 2x_4 &= 4 \end{aligned}\right\}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 4 & -1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & -2 & 4 \end{array}\right] \xrightarrow[R3' = R3 - R1]{R2' = R2 - R1} \left[\begin{array}{cccc|c} \boxed{1} & 0 & -1 & 4 & -1 \\ 0 & \boxed{1} & 2 & -4 & 2 \\ 0 & 1 & 3 & -6 & 5 \end{array}\right] \xrightarrow{R3'' = R3' - R2'}$$

$$\underbrace{\left[\begin{array}{cccc|c} \boxed{1} & 0 & -1 & 4 & -1 \\ 0 & \boxed{1} & 2 & -4 & 2 \\ 0 & 0 & \boxed{1} & -2 & 3 \end{array}\right]}_{\text{Echelon form}} \xrightarrow[R1' = R1 - R3'']{R2'' = R2' - R3''} \underbrace{\left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 2 & 2 \\ 0 & \boxed{1} & 0 & 0 & -4 \\ 0 & 0 & \boxed{1} & -2 & 3 \end{array}\right]}_{\text{Row Reduced Echelon Form}}$$

Therefore,

$$\left.\begin{aligned} x_4 &= \alpha \in \mathbb{R} \text{ free variable,} \\ x_1 &= 2 - 2\alpha \\ x_2 &= -4 \\ x_3 &= 3 + 2\alpha \end{aligned}\right\} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 3 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Example: Say that a system $A\vec{x}=\vec{b}$ is transform through Gauss-Jordan elim. into $R\vec{x}=\vec{d}$, with R row reduced ech. form of A. Find R and $\vec{d}$ if the complete solution is

$$\vec{x}=\begin{bmatrix}4\\0\\1\\0\end{bmatrix}+\alpha\begin{bmatrix}1\\1\\0\\0\end{bmatrix}+\beta\begin{bmatrix}-1\\0\\2\\1\end{bmatrix} \qquad (A \text{ is } 4\times 4)$$

Sol:

There are two parameters $\to$ 2 free variables $\to$ 2 pivots.

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \end{array}$$
$$\begin{bmatrix}\boxed{1} & -1 & 0 & 1\\ 0 & 0 & \boxed{1} & -2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \qquad \vec{d}=\begin{bmatrix}4\\1\\0\\0\end{bmatrix}$$

$\uparrow$ $\uparrow$
free variables

3) Inverse

Def: The matrix A has an inverse, denoted A^{-1}, if

$$AA^{-1}=A^{-1}A=I.$$

Question: Which matrices have an inverse?

Let's try an example:

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix} \to A^{-1}? \quad \text{Denote } A^{-1} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}, \quad x_{11}, x_{12}, x_{21}, x_{22}?$$

Use definition: $AA^{-1} = I \Rightarrow \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$

$$\Rightarrow \begin{bmatrix} 4x_{11} + 2x_{21} & 4x_{12} + 2x_{22} \\ x_{11} + x_{21} & x_{12} + x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} \begin{bmatrix} 4x_{11} + 2x_{21} \\ x_{11} + x_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 4x_{12} + 2x_{22} \\ x_{12} + x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases} \Bigg\} \text{ Two linear systems.}$$

So now we can solve this systems to find $x_{11}, \dots, x_{22}$.

If one of the systems doesn't have a solution, then the inverse doesn't exist.

Notice that we would solve them using the augmented matrices

$$\left[\begin{array}{cc|c} 4 & 2 & 1 \\ 1 & 1 & 0 \end{array}\right], \quad \left[\begin{array}{cc|c} 4 & 2 & 0 \\ 1 & 1 & 1 \end{array}\right]$$

so we can solve them together $\left[\begin{array}{cc|cc} 4 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$

• Method to find A^{-1}: Gauss-Jordan.

1) Create the augmented matrix $[A \quad I]$

Ex: $A = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix} \rightsquigarrow \left[\begin{array}{cc:cc} 4 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$

2) Apply elimination to get echelon form:

$$[A \quad I] \rightsquigarrow \begin{bmatrix} \ast & \ast & \cdots \\ 0 & \ast & \cdots \end{bmatrix} \qquad \begin{bmatrix} 4 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 2 & 1 & 0 \\ 0 & 1/2 & -1/4 & 1 \end{bmatrix}$$

3) Are there n pivots? → Yes: continue
→ No: The inverse does not exist.

4) Continue until obtain the identity on the left:

$$\begin{bmatrix} 4 & 2 & 1 & 0 \\ 0 & 1/2 & -1/4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 2 & 1 & 0 \\ 0 & 2 & -1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 0 & 2 & -4 \\ 0 & 2 & -1 & 4 \end{bmatrix} \rightarrow$$

$$\rightarrow \begin{bmatrix} 1 & 0 & \underbrace{\begin{matrix} 1/2 & -1 \\ -1/2 & 2 \end{matrix}}_{A^{-1}} \\ 0 & 1 & \end{bmatrix} \rightarrow A^{-1} = \begin{bmatrix} 1/2 & -1 \\ -1/2 & 2 \end{bmatrix}$$

<u>Remark</u>: A^{-1} exists if A is row equivalent to the identity.

~~Remark~~.

Properties:

1) $(A^{-1})^{-1} = A$

2) If A,B invertible, $(AB)^{-1} = B^{-1}A^{-1}$

3) $(A+B)^{-1} \neq A^{-1} + B^{-1}$

4) If the inverse exists, it is unique.

5) If $A\vec{x} = \vec{0}$ for some $\vec{x} \neq \vec{0}$, then A^{-1} doesn't exist.

Proofs:

1) Let $B = A^{-1}$. Is $B^{-1} = A$?

$BA = AB = I$? ✓

2) ~~Let~~ $\cancel{(AB)^{-1} = \quad (AB)}$
Let $C = B^{-1}A^{-1}$. Is $C(AB) = (AB)C = I$?

4) Assume $B \neq C$ two inverses of A. Then

$$\left.\begin{aligned} B(AC) &= BI = B \\ (BA)C &= IC = C \end{aligned}\right\} \Rightarrow B = C \;\lightning.$$

5) Assume A^{-1} exists. Then, $A^{-1}A\vec{x} = A^{-1}\vec{0} \Rightarrow I\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0} \rightarrow\!\leftarrow$.

4] Elementary matrices and elimination.

Gauss-Jordan: $[A \quad I] \underset{\text{operation}}{\overset{\text{row}}{\rightsquigarrow}} [I \quad A^{-1}]$

Same as $[A \quad I] \underset{A^{-1}}{\rightsquigarrow} [I \quad A^{-1}]$

That is, row operations can be encoded as multiplication by certain matrices.

- Three types of row operations:

T1 : Substract l_{ij} times equation j from equation i.

T2 : Interchange row i and row j.

T3 : Scale row i by some number

Which matrices represent these operations?

1) <u>T1 operation</u>

Consider $A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ and let's do first step in elimination:

$R2' = R2 - 2R1$

$$\begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \\ 1 & 0 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 1 & 0 \\ 0 & -2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

We want

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}} \underbrace{\begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \\ 1 & 0 & 3 \end{bmatrix}}_{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

In general, the matrix for a TI operation is the identity with an extra nonzero entry given by $-l_{ij}$ in the (i,j) position.

Ex: Write all the steps as elementary matrices

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -2 \\ 2 & -3 & 3 \end{bmatrix} \xrightarrow[\cancel{R3' = R2}]{R2' = R2 - R1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -3 \\ 2 & -3 & 3 \end{bmatrix} \xrightarrow{R3' = R3 - 2R1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -3 \\ 0 & -5 & 1 \end{bmatrix} \rightarrow$$

$$\xrightarrow{R3'' = R3' - \frac{-5}{-3} R2'} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & 6 \end{bmatrix}$$

So

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5/3 & 1 \end{bmatrix}$$

Remark: $A \xrightarrow{E_{21}} \xrightarrow{E_{31}} \xrightarrow{E_{32}} U$ (upper triangular), i.e.

$$U = E_{32} E_{31} E_{21} A.$$

• Q: Can we always invert these E_{ij} matrices?

Of course:

E_{ij}: subtracts l_{ij} times eq. j from eq. i.

E_{ij}^{-1} does the opposite effect: adds l_{ij} times eq. j ~~from~~ to eq. i.

So,

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \ldots$$

2) T2 operation:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P_{23}} \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 3 \\ 4 & 0 & 1 \end{bmatrix}$$

Matrix P_{ij} = identity with rows i and j interchanged.

→ of course, $P_{ij}^{-1} = P_{ij}$ ✓.

3) T3: operation

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 2 & 4 & 0 \\ 4 & 0 & 1 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 0 \\ 12 & 0 & 3 \\ 1 & 0 & 3 \end{bmatrix} \leftarrow 2^{nd} \text{ eq. multiplied by } 3.$$

"identity matrix with row i multiplied by λ".